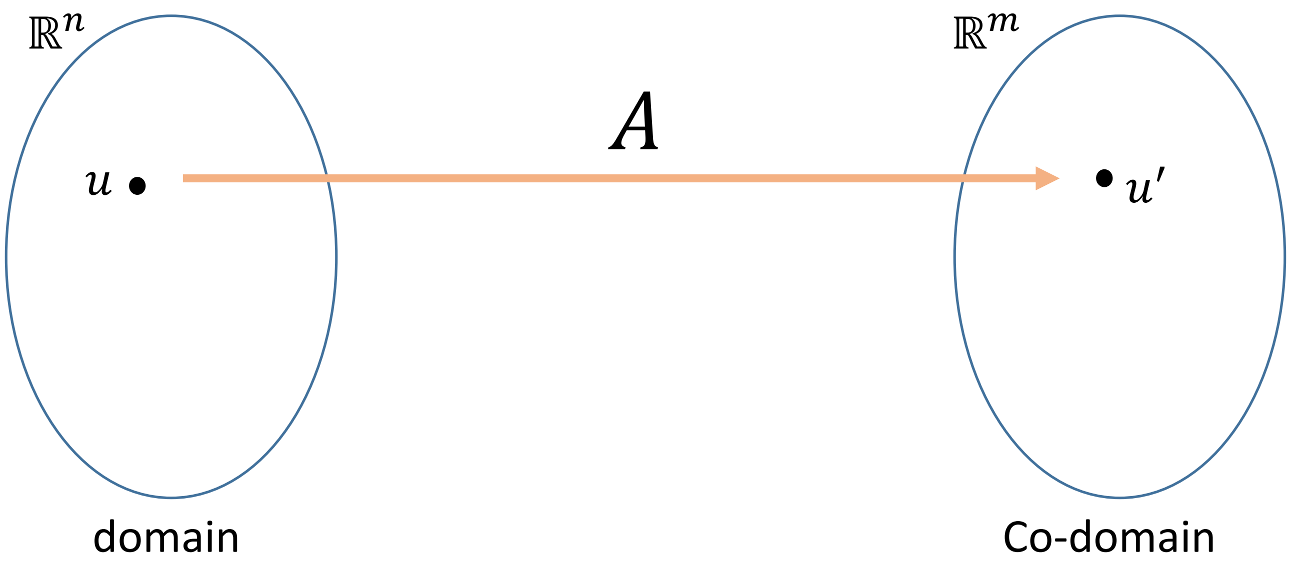
Linear Transformations in Graphics

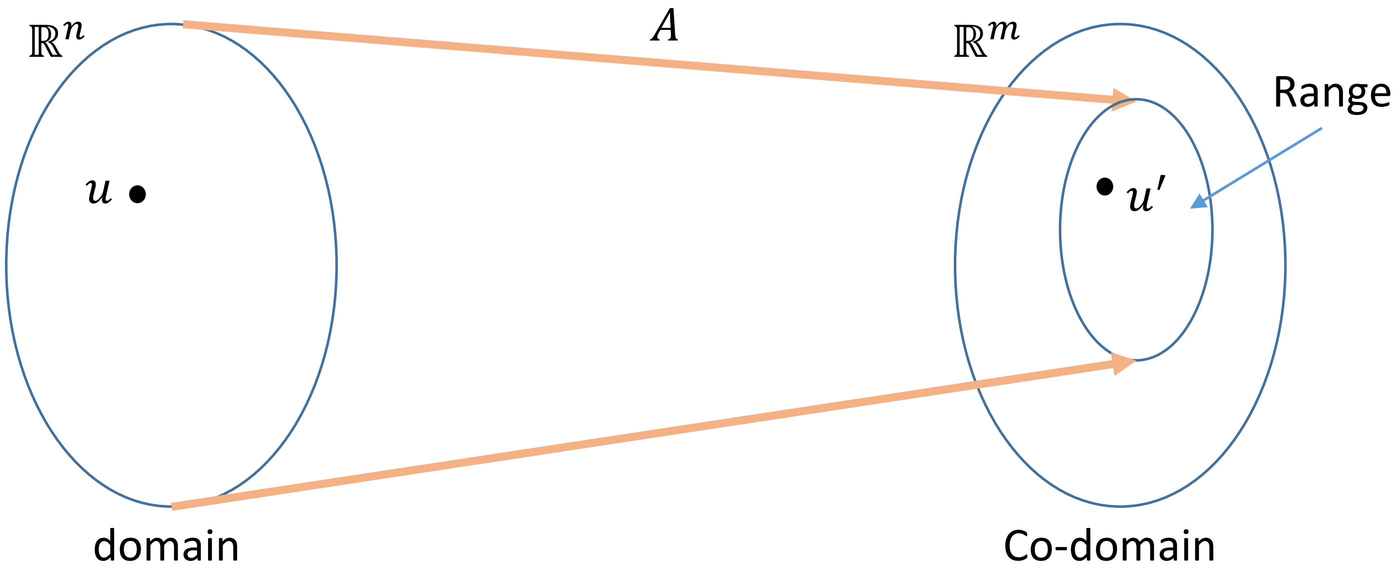
Linear Transformation

- Matrix-vector multiplication is a linear transformation.
- If A is a $m \times n$ matrix, and u is $n \times 1$ vector, then matrix A maps the vector u from $\mathbb{R}^n$ to the $m \times 1$ vector u' in $\mathbb{R}^m$.



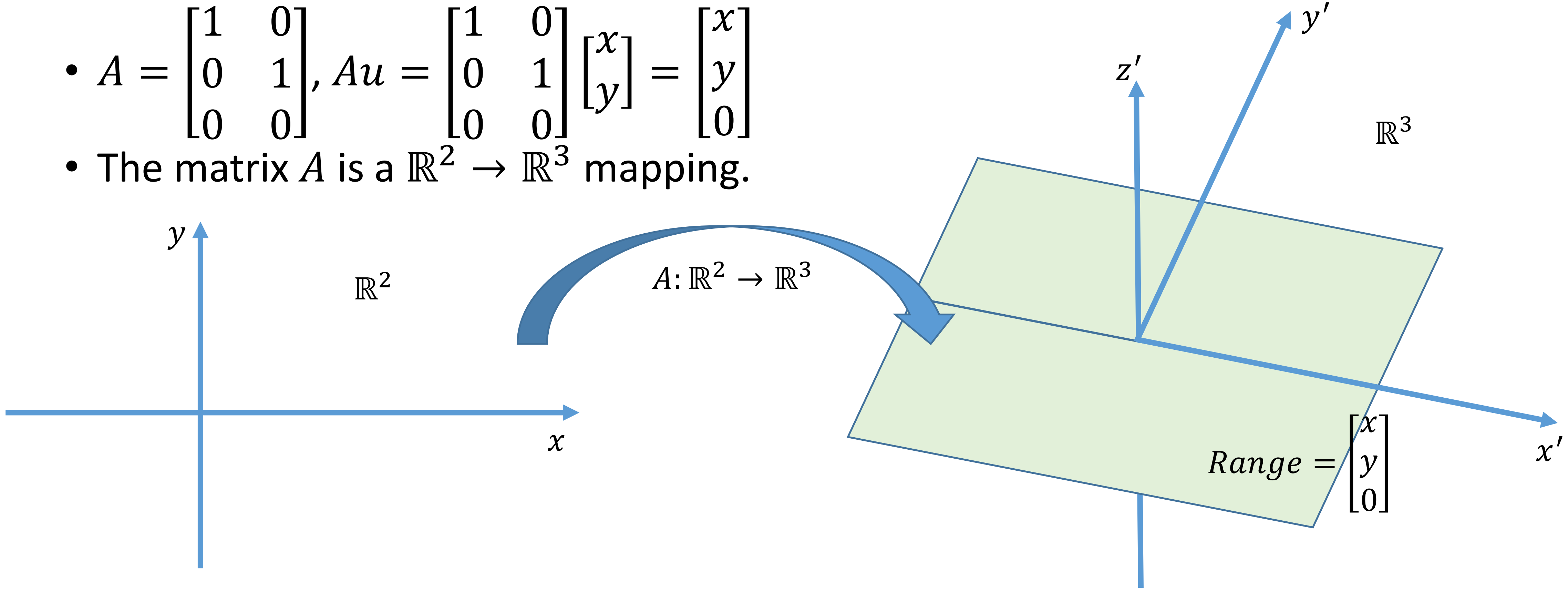
Linear Transformation

- The space in which the vector u exist is called "Domain" ($\mathbb{R}^n$).
- The space in which the vector u is mapped to is called "Co-domain" ($\mathbb{R}^m$).
- The subset of co-domain in which all $\mathbb{R}^n$ vectors are mapped into is called "Range".



Linear Transformation

- Example:
  - $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $Au = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$
  - The matrix A is a $\mathbb{R}^2 \to \mathbb{R}^3$ mapping.



Linearity condition

- A transformation T is linear if it meets these requirements for all u and v:

$$T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$$
$$T(c\boldsymbol{u}) = cT(\boldsymbol{u})$$

- From above implied that

$$T(\boldsymbol{0}) = \boldsymbol{0}$$
$$T(c\boldsymbol{u} + d\boldsymbol{v}) = cT(\boldsymbol{u}) + dT(\boldsymbol{v})$$

Linear transformation theorem

- Let T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then there exist a unique matrix A such that
$$T(x) = Ax \quad for\ all\ x\ in\ \mathbb{R}^n$$
- A is a $m \times n$ matrix.
- jth column of A is the vector $T(e_j)$, where e_j is the jth column of the identity matrix in $\mathbb{R}^n$:
$$A = [T(e_1) \quad \ldots \quad T(e_n)]$$

Linear transformation theorem

- Proof: Write $x = I_n x = [e_1 \quad \dots \quad e_n]x = x_1 e_1 + \cdots + x_n e_n$, and use the linearity of T to compute

$$T(x) = T(x_1 e_1 + \cdots + x_n e_n) = x_1 T(e_1) + \cdots + x_n T(e_n)$$

$$= [T(e_1) \quad \dots \quad T(e_n)] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \mathrm{Ax}$$

Linear Transformation

- Example: Let $T\colon\ \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle φ, with counterclockwise rotation for a positive angle.
- Such a transformation is linear.
- $\begin{bmatrix}1\\0\end{bmatrix}$ rotates into $\begin{bmatrix}\cos\varphi\\ \sin\varphi\end{bmatrix}$, and $\begin{bmatrix}0\\1\end{bmatrix}$ rotates into $\begin{bmatrix}-\sin\varphi\\ \cos\varphi\end{bmatrix}$
- By linear transformation theorem

$$A = \begin{bmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{bmatrix}$$

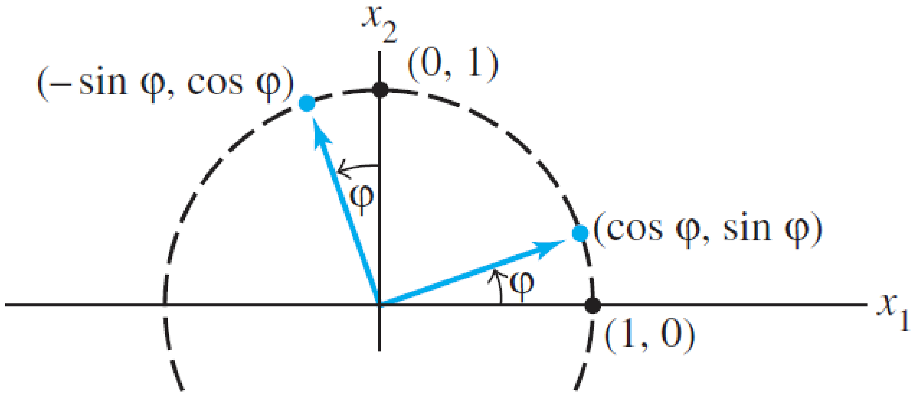
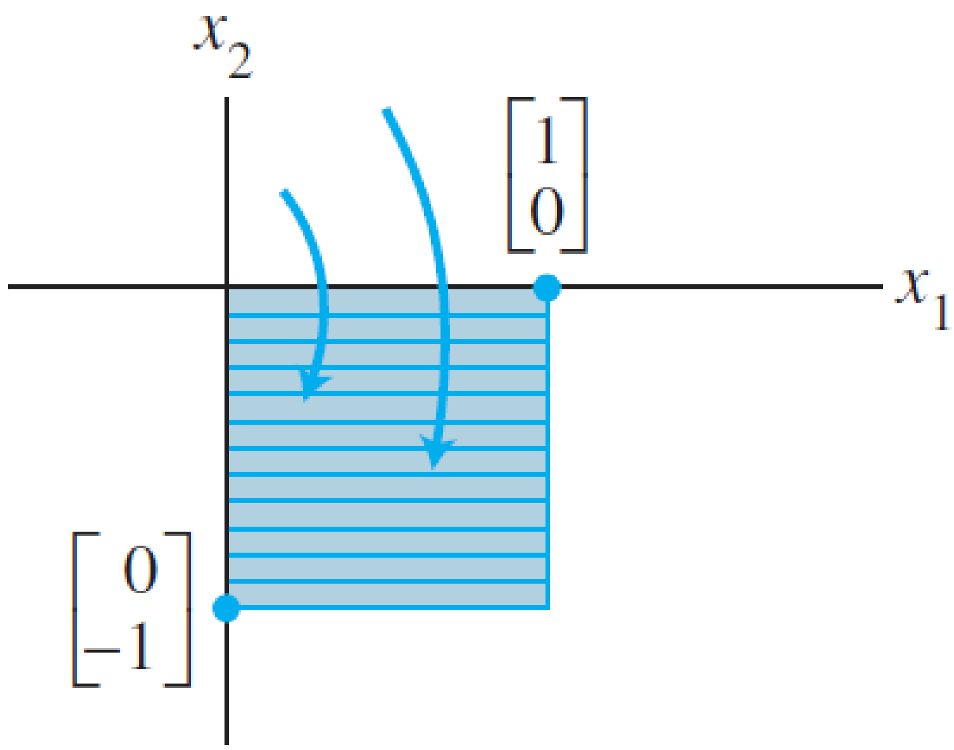


FIGURE 1 A rotation transformation.

Geometric Transformations

- Reflection Through the x_1-axis
- Matrix:

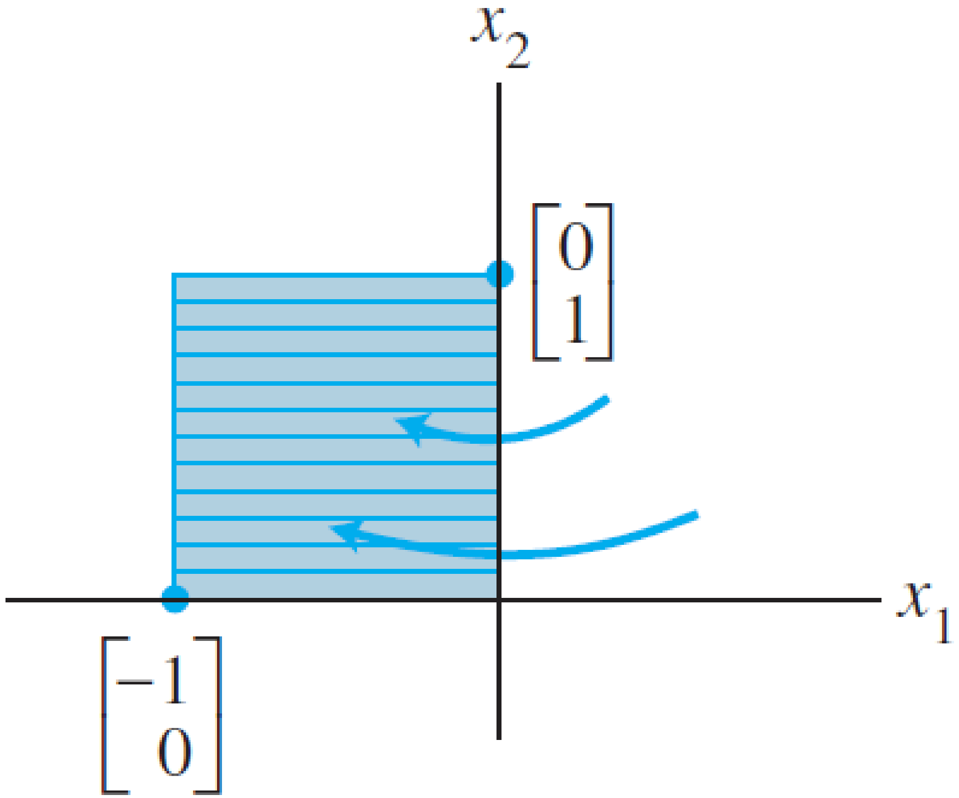
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



Geometric Transformations

- Reflection Through the x_2-axis
- Matrix:

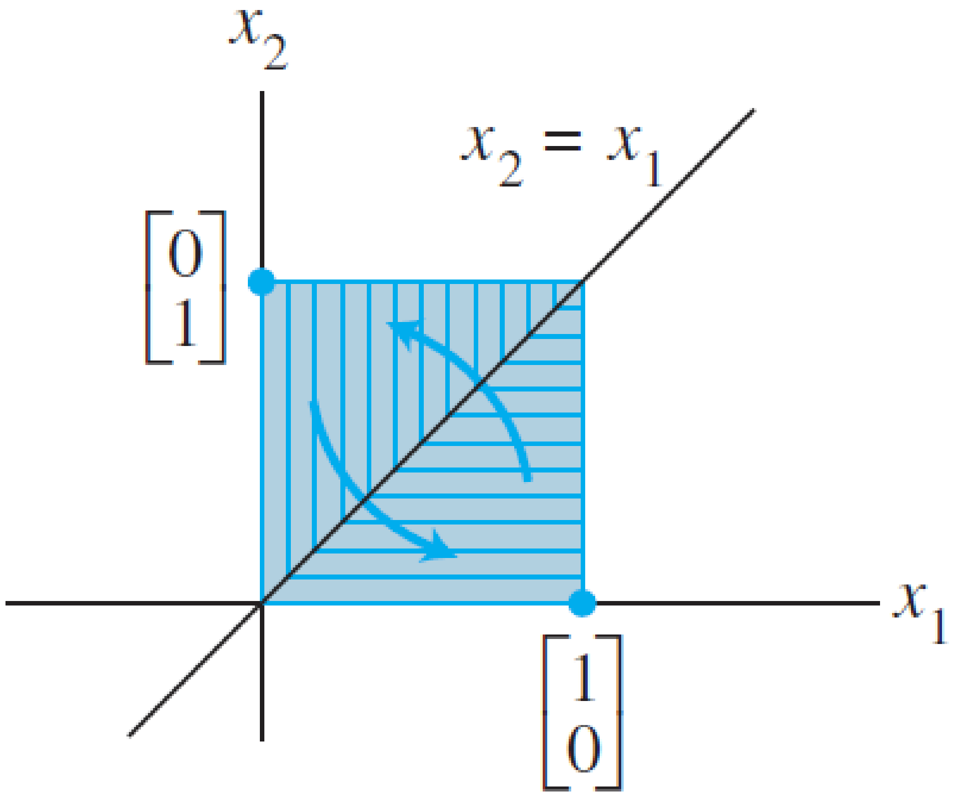
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



Geometric Transformations

- Reflection Through the line $x_2 = x_1$
- Matrix:

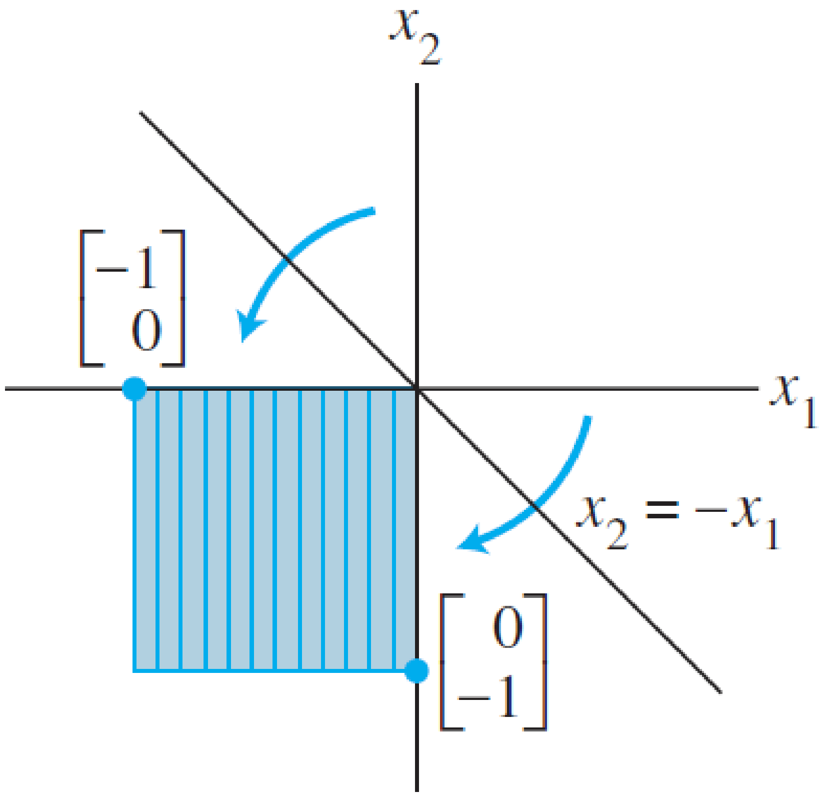
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



Geometric Transformations

- Reflection Through the line $x_2 = -x_1$
- Matrix:

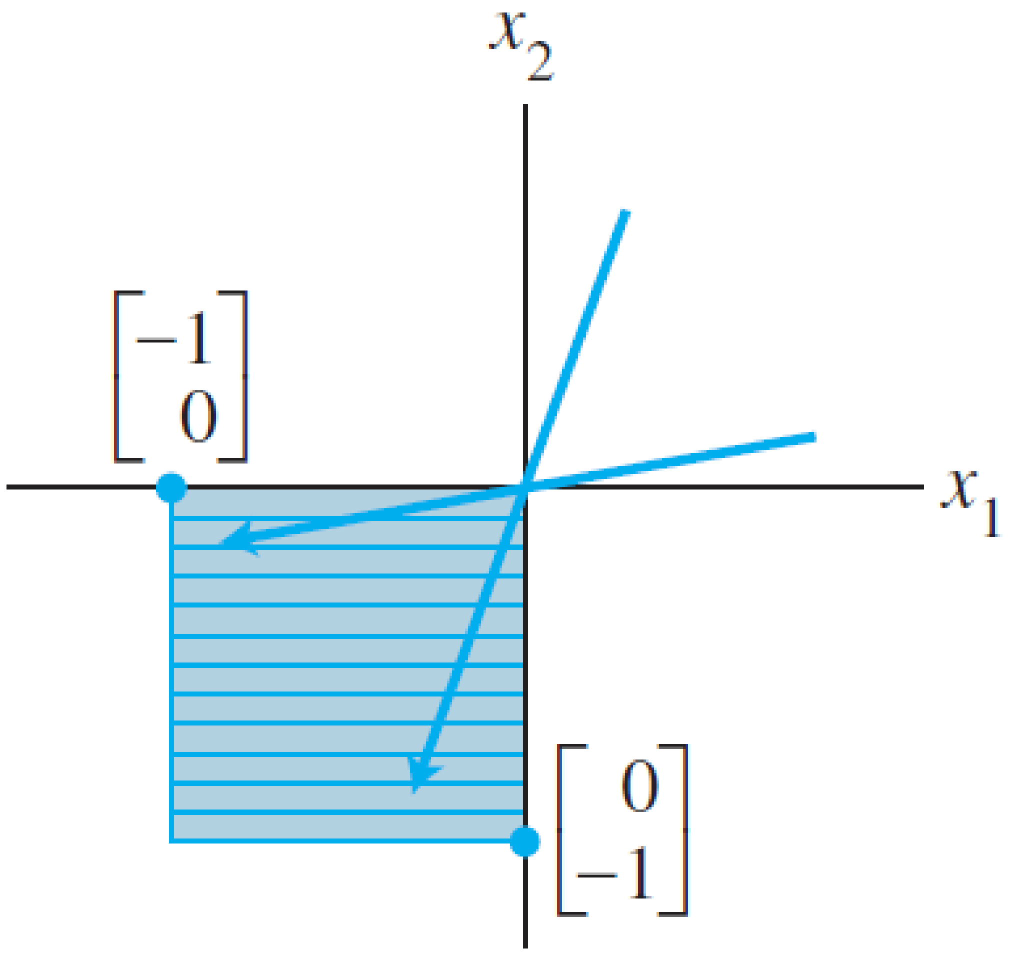
$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$



Geometric Transformations

- Reflection Through the origin
- Matrix:

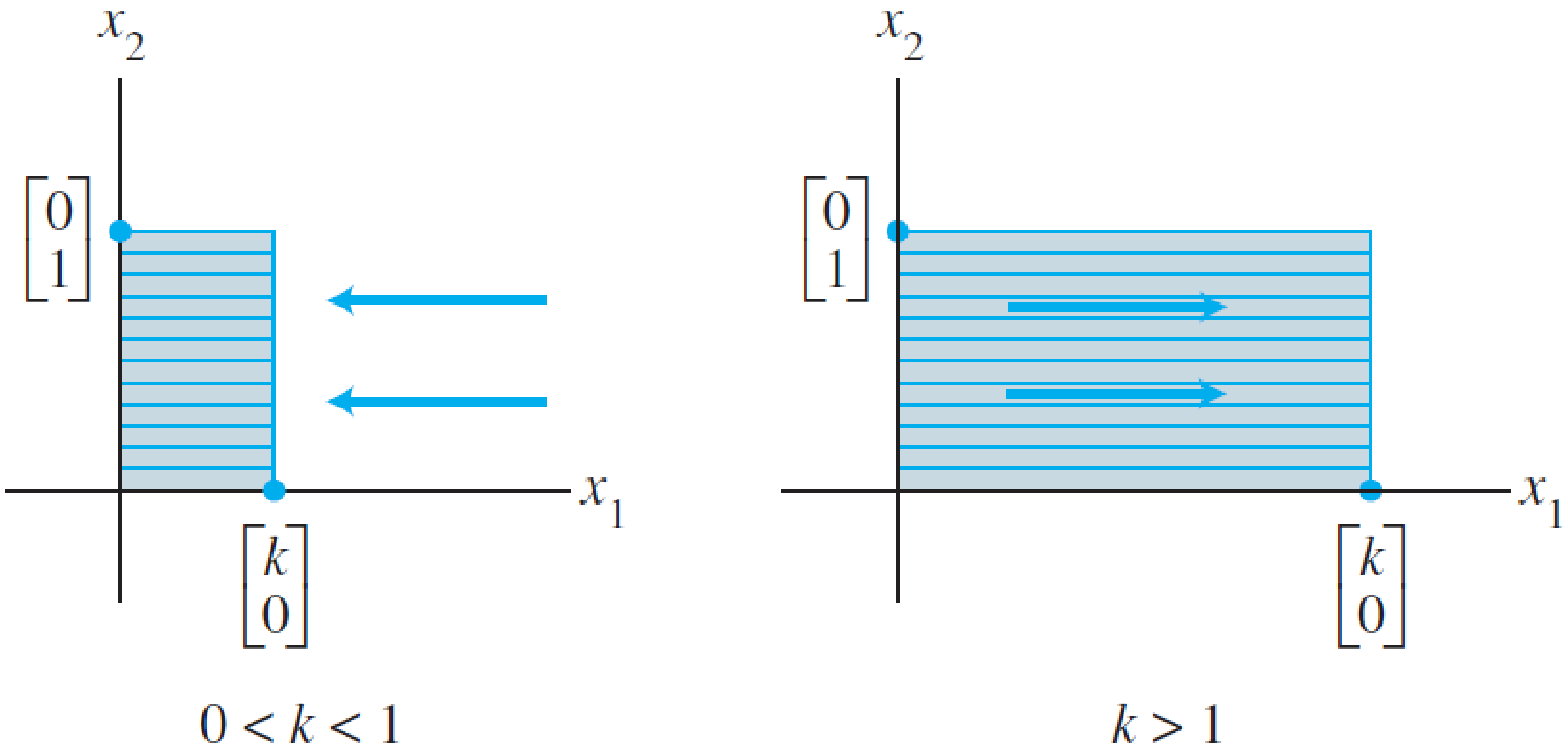
$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$



Geometric Transformations

- Horizontal contraction and expansion
- Matrix:

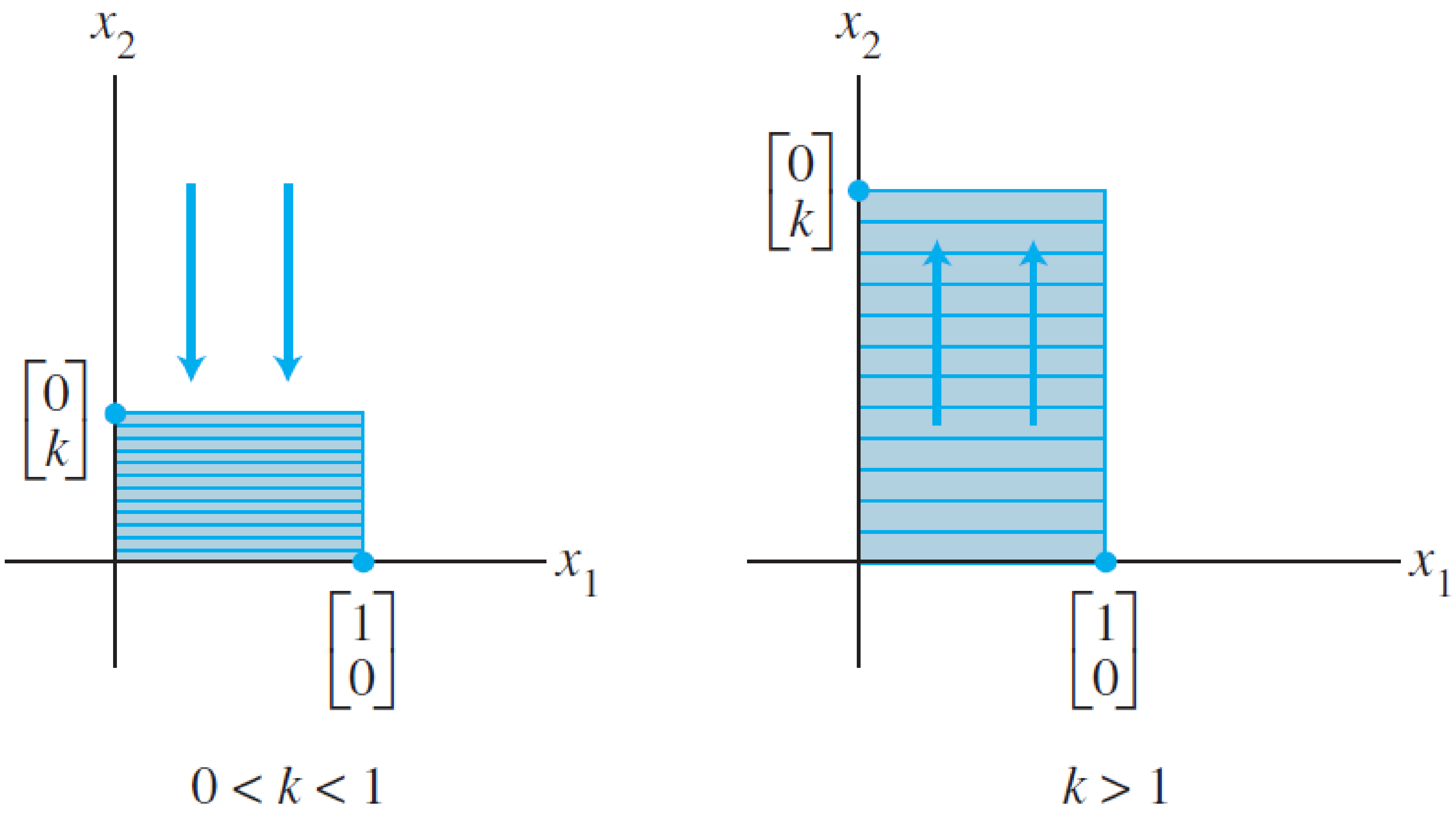
$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$



Geometric Transformations

- Vertical contraction and expansion
- Matrix:

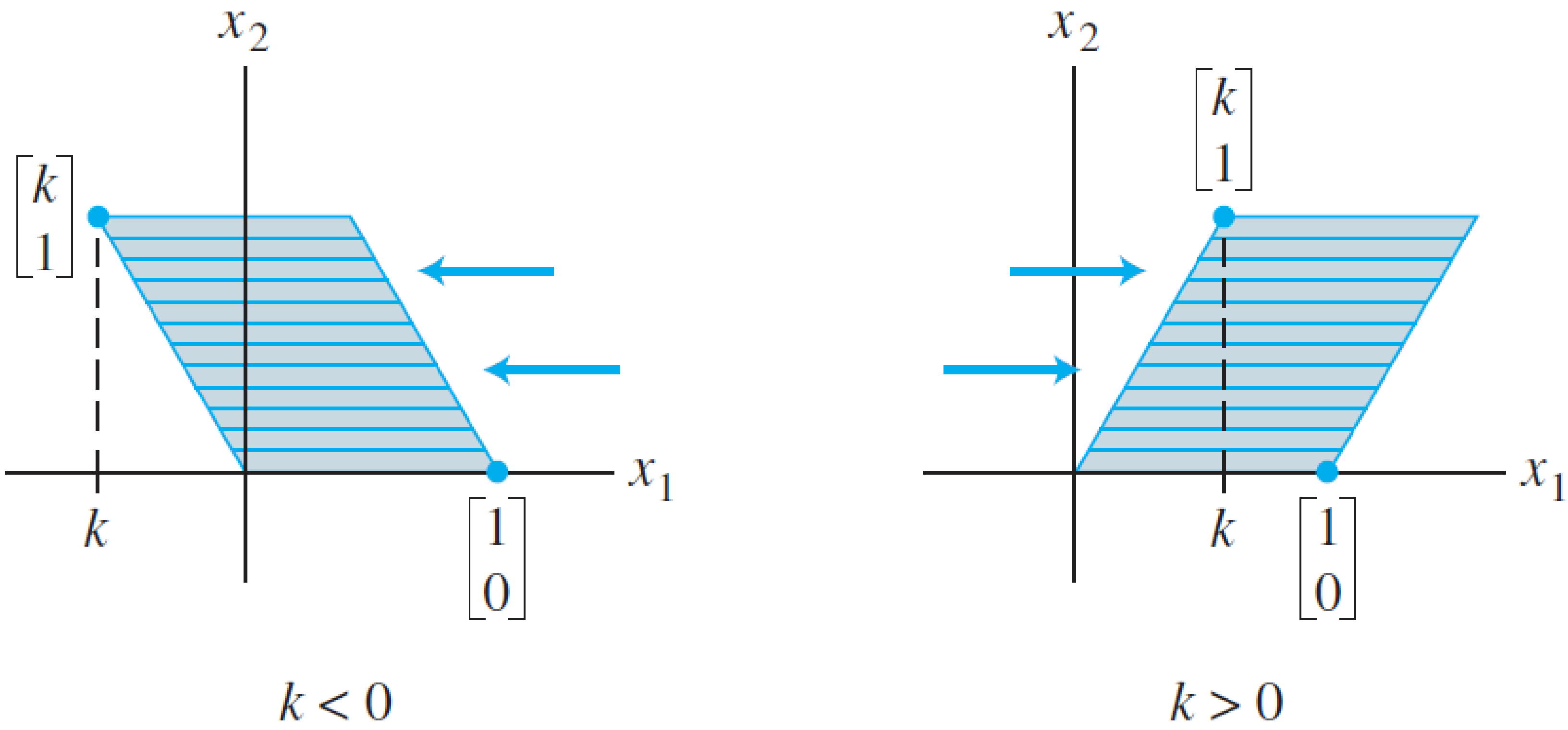
$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$



Geometric Transformations

- Horizontal shear
- Matrix:

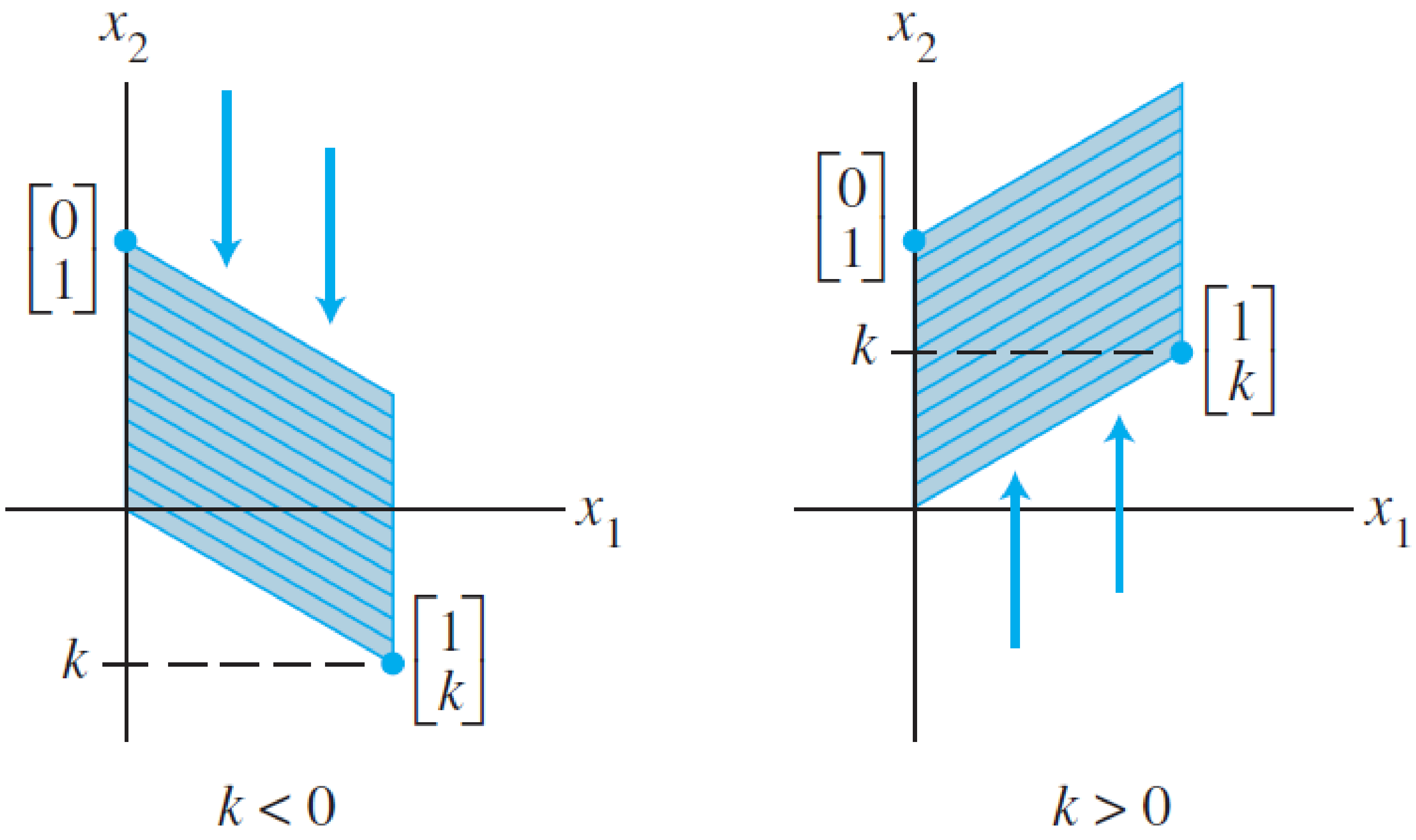
$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$



Geometric Transformations

- Vertical shear
- Matrix:

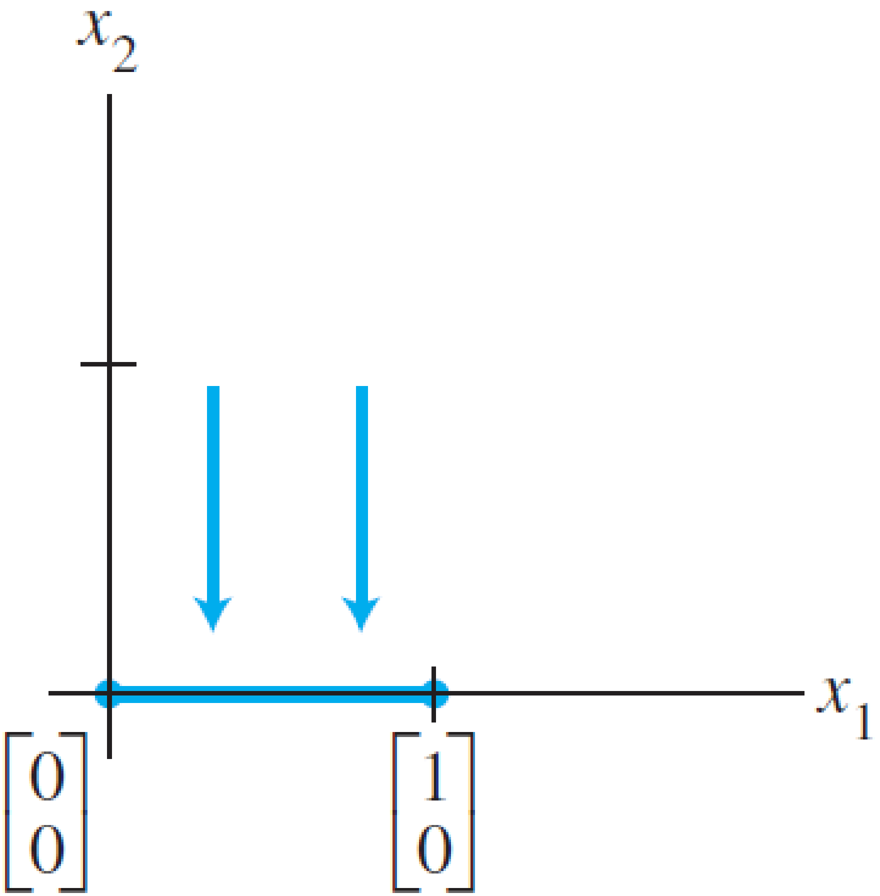
$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$



Geometric Transformations

- Projection onto the x_1-axis
- Matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



Geometric Transformations

- Projection onto the x_2-axis
- Matrix:

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$