Symmetric Matrices and Quadratic Forms

Quadratic form

- Suppose $\boldsymbol{x}$ is a column vector in $\mathbb{R}^n$, and A is a symmetric $n \times n$ matrix.
- The term $\boldsymbol{x}^T A \boldsymbol{x}$ is called a quadratic form.
- The result of the quadratic form is a scalar. $(1 \times n)(n \times n)(n \times 1)$
- The quadratic form is also called a quadratic function $Q(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x}$.
- The quadratic function's input is the vector x and the output is a scalar.

Quadratic form

- Suppose x is a vector in $\mathbb{R}^3$, the quadratic form is:
- $Q(\boldsymbol{x}) = \boldsymbol{x}^T A\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$
- $Q(\boldsymbol{x}) = a_{11}{x_1}^2 + a_{22}{x_2}^2 + a_{33}{x_3}^2 + \cdots$

$$(a_{12} + a_{21})x_1x_2 + (a_{13} + a_{31})x_1x_3 + (a_{23} + a_{32})x_2x_3$$

- Since A is symmetric $a_{ij} = a_{ji}$, so:
- $Q(\boldsymbol{x}) = a_{11}{x_1}^2 + a_{22}{x_2}^2 + a_{33}{x_3}^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$

Quadratic form

- Example: find the quadratic polynomial for the following symmetric matrices:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

  - $Q(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = {x_1}^2 + 2{x_2}^2$

  - $Q(\boldsymbol{x}) = \boldsymbol{x}^T B \boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = {x_1}^2 + 2{x_2}^2 - {x_3}^2 - 2x_1x_2 + 2x_2x_3$

Motivation for quadratic forms

- Example: Consider the function

$$Q(x) = 8x_1^2 - 4x_1x_2 + 5x_2^2$$

Determine whether Q(0,0) is the global minimum.

- Solution we can rewrite following equation as quadratic form

$$Q(x) = x^T A x \quad where \quad A = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$$

The matrix A is symmetric by construction. Eigen vectors of A are

$$v_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \qquad v_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

With associated eigenvalues $\lambda_1 = 9\ and \lambda_2 = 4$

Example Cont'

- Let $x = c_1 v_1 + c_2 v_2$, Now we have

$$Q(x) = x^T A x = \lambda_1 c_1^2 + \lambda_2 c_2^2 = 9c_1^2 + 4c_2^2$$

Therefore Q(x)>0 and Q(0,0) is the global minimum.

Quadratic form

- Example: find the symmetric matrix for the following quadratic polynomials:

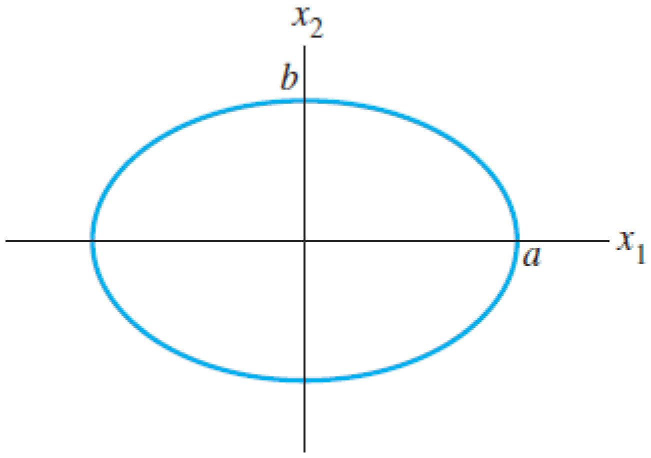
$$Q_1(\boldsymbol{x}) = {x_1}^2 + {x_2}^2 + 2x_1x_2$$

$$Q_2(\boldsymbol{x}) = {x_1}^2 + {x_2}^2 + 2x_1x_2 - 4x_1x_3 + 2\sqrt{2}x_2x_3$$

  - $Q_1(\boldsymbol{x}) = \boldsymbol{x}^T A\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
  - $Q_2(\boldsymbol{x}) = \boldsymbol{x}^T B\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & \sqrt{2} \\ -2 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

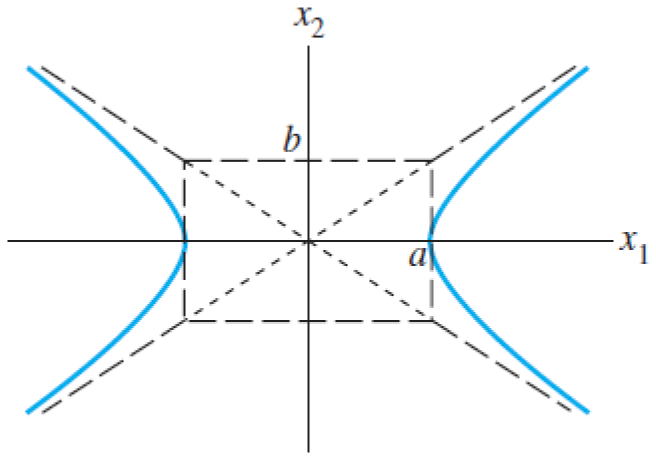
Geometric interpretation in $\mathbb{R}^2$

- For diagonal matrix A, $Q(\boldsymbol{x})$ is an ellipse, or hyperbola, or intersection of two lines, or a point.



$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1, \ a > b > 0$$

ellipse



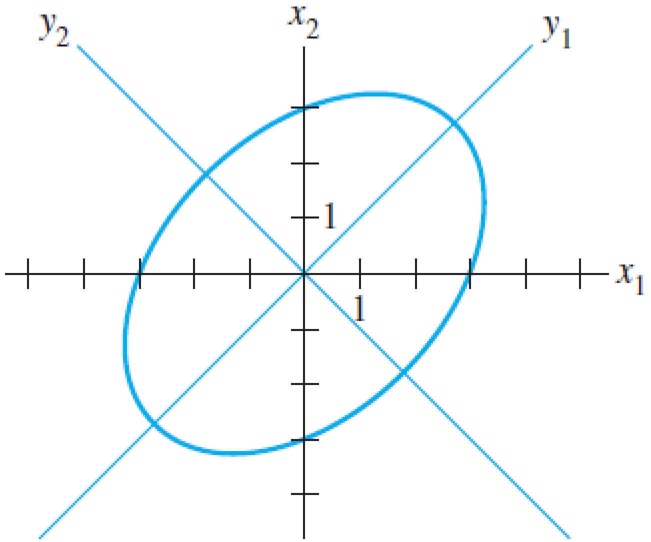
$$\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1, \ a > b > 0$$

hyperbola

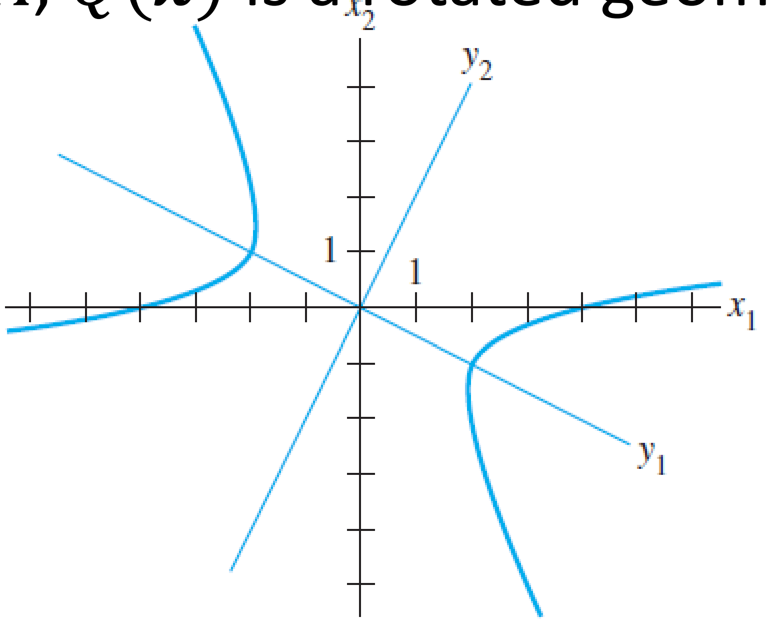
Motivation: This form happens for **diagonal matrices** and maxima and minima appear along the eigenvectors and a and b are the eigenvalues

Geometric interpretation in $\mathbb{R}^2$

- For non-diagonal matrix A, $Q(\boldsymbol{x})$ is a rotated geometry.



(a) $5x_1^2 - 4x_1x_2 + 5x_2^2 = 48$



(b) $x_1^2 - 8x_1x_2 - 5x_2^2 = 16$

Motivation: This form happens for **nondiagonal matrices** and maxima and minima appear along the eigenvectors (but not aligned) . For aligning them we can use change of variables as explained in next slides

Change of variable

- We can convert the rotated ellipse or hyperbola to its standard form.
- Recall that we can diagonalize symmetric matrices.
- If A is a symmetric matrix, it can be diagonalized as $D = P^T AP$, where P is the orthogonal matrix of eigenvectors of A.
- Suppose $\boldsymbol{x} = P\boldsymbol{y}$, then

$$Q(\boldsymbol{x}) = \boldsymbol{x}^T A\boldsymbol{x} = (P\boldsymbol{y})^T A(P\boldsymbol{y}) = \boldsymbol{y}^T P^T AP\boldsymbol{y} = \boldsymbol{y}^T D\boldsymbol{y}$$

Motivation: Alignment is required for change of variable. This is diagonalization of symmetric matrices

Change of Variable Example

- Example: Make a change of variable that transforms the quadratic form $Q(x) = x_1^2 - 8x_1x_2 - 5x_2^2$ into a quadratic form with no cross-product term.
- Solution if we write Q(x) as quadratic form, matrix A is

$$A = \begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix}$$

The first step is to orthogonally diagonalize A. Its eigenvalues and with associated unit eigenvectors are

$$\lambda = 3: \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}; \qquad \lambda = -7: \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

Example Cont'

- Then $A = PDP^{-1}$ and $D = P^{-1}AP = P^{T}AP$. A suitable change of variable is

$$\mathbf{x} = P\mathbf{y}, \qquad \text{where } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Then

$$\begin{aligned} x_1^2 - 8x_1x_2 - 5x_2^2 &= \mathbf{x}^TA\mathbf{x} = (P\mathbf{y})^TA(P\mathbf{y}) \\ &= \mathbf{y}^TP^TAP\mathbf{y} = \mathbf{y}^TD\mathbf{y} \\ &= 3y_1^2 - 7y_2^2 \end{aligned}$$