Lecture 2: Vector-Vector Operations

- Vector-Vector Operations
  - Addition of two vectors
    - Geometric representation of addition and subtraction of vectors
    - Vectors and points
  - Dot product of two vectors
    - Geometric interpretation of the dot product of two vectors
    - Computation of Dot product
    - Dot product of perpendicular vectors
    - Dot product of a vector with itself
    - Examples: Decomposition of force vectors, Decomposition of a vector into orthogonal components, coordinates of a point in an orthogonal coordinate system.
  - Cross product of two three dimensional vectors (Self-study)
    - Geometric interpretation of a cross product
      - Area of a triangle
    - Cross product of orthogonal and parallel vectors
  - Scalar Triple Product (Self-study)
    - Geometric interpretation of a scalar triple product
      - Volume of a parallelepiped

Vector-Vector Operations

- Vector addition (and subtraction)
  - $\boldsymbol{a} + \boldsymbol{b}, \boldsymbol{a} - \boldsymbol{b}$
- Vector Multiplication
  - Dot Product: $\boldsymbol{a} \cdot \boldsymbol{b}$
  - Cross Product: $\boldsymbol{a} \times \boldsymbol{b}$

Vector Addition (Page 26)

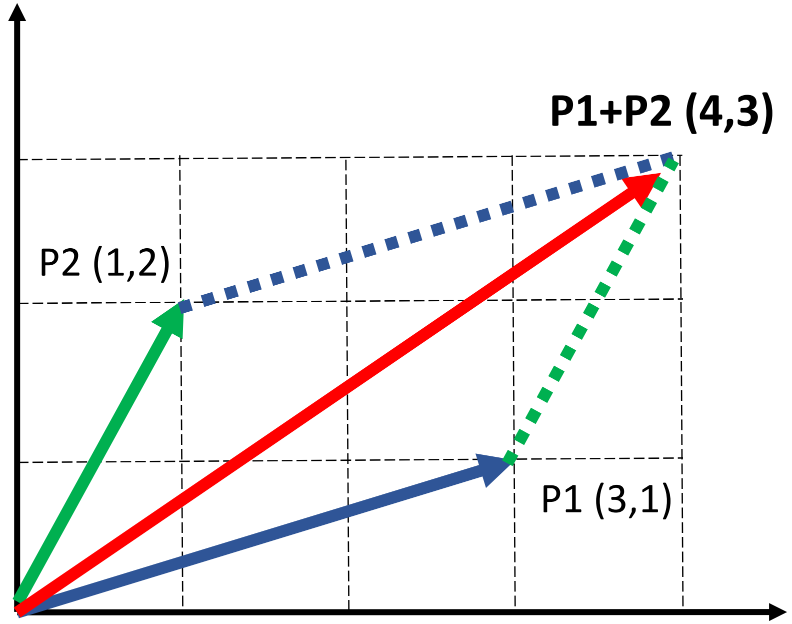
- Two vector can be added only if they have the same dimension.
- The corresponding components of the two vector are added together.
- Two vector can be subtracted in the same way of adding, by subtracting components.
- Example:
  - $\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$
  - $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} \sqrt{2} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-\sqrt{2} \\ 0-1 \\ -1-0 \end{bmatrix} = \begin{bmatrix} 1-\sqrt{2} \\ -1 \\ -1 \end{bmatrix}$

When can two vectors be added?

- Only if two vectors have the same dimension they can be added.
- Row vectors and column vectors of the same dimension can be added.
- Example:
  - $[1] + [0] = [1]$
  - $\begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$
  - $\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0.5 \\ 1 \end{bmatrix}$ = ?

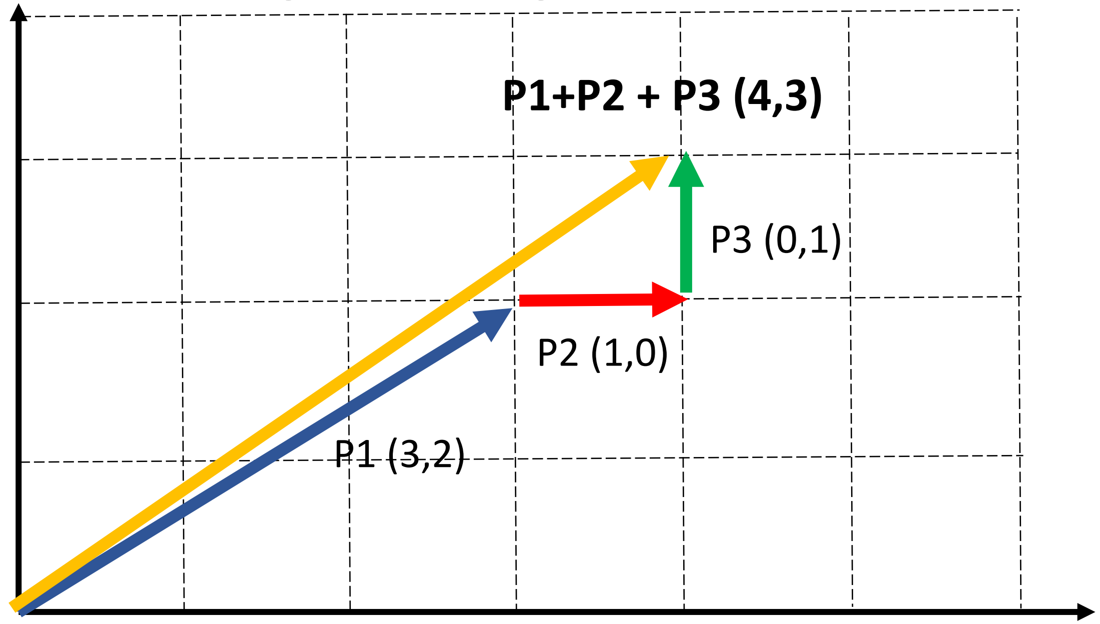
How to represent addition of two vectors graphically (Ch. 1.3 P. 26)

- Draw one vector.
- Draw the other vector.
- Draw one vector along the diagonal of the parallelogram formed by P1 and P2.
- Example:
  - $\begin{bmatrix}3\\1\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}4\\3\end{bmatrix}$



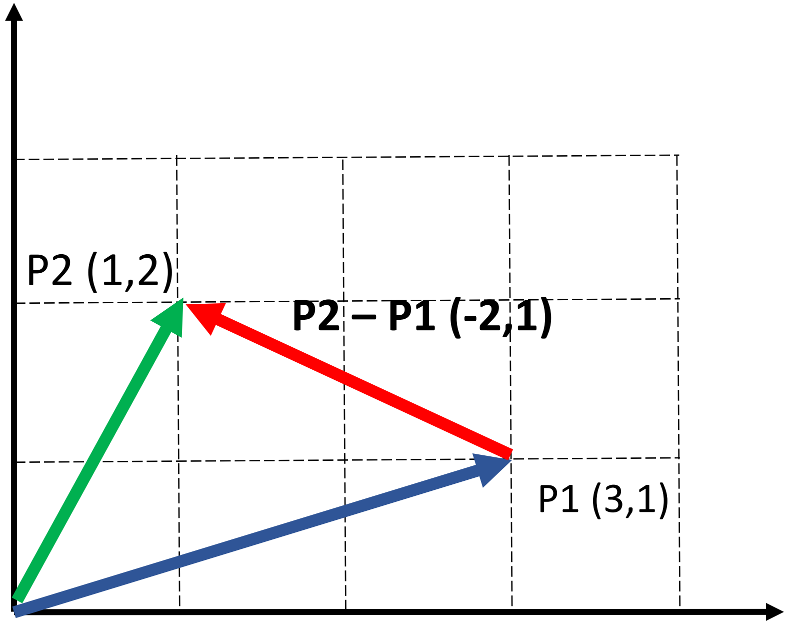
How to represent addition of three vectors graphically

- Another way to add multiple vectors graphically is to link the tail of one vector with the head of another vector as shown below.
- The final vector is obtained by connecting the origin and the head of the last vector.
- Example:
  - $\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}4\\3\end{bmatrix}$



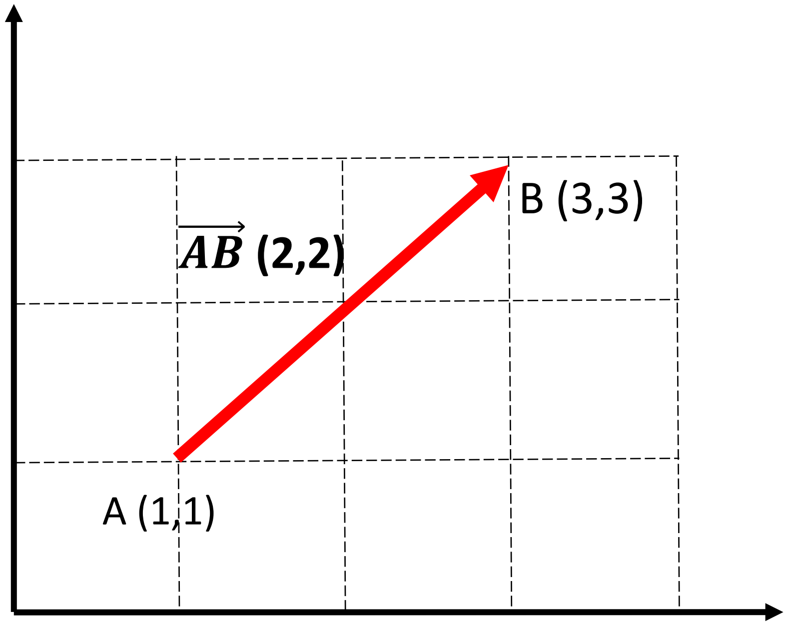
How to represent subtraction of two vectors graphically (Ch. 1.3 P. 26)

- Draw one vector.
- Draw the other vector.
- Draw one vector originated at the tail of the first vector, and ends at the tail of the second vector.
- Example:
  - $\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}-2\\1\end{bmatrix}$



How to compute vector between points?

- Given two points $P_1 = (x_1, y_1, z_1)$and $P_2 = (x_2, y_2, z_2)$
- The vector to P_1 from the origin is $\boldsymbol{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and vector to P_2 from the origin is $\boldsymbol{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$
- The vector from P_1 to P_2 is $\boldsymbol{v}_2$-$\boldsymbol{v}_1 = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}$



Algebraic Properties of vector addition and subtraction

Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$,
where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(vii) $c(d\mathbf{u}) = (cd)(\mathbf{u})$

(viii) $1\mathbf{u} = \mathbf{u}$

What is a dot product (Inner product)?

- Dot product or Inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is represented as:
  - $\boldsymbol{a} \cdot \boldsymbol{b} = s$
- Dot product of two vectors results in a scalar.
- Multiply the corresponding components of the two vectors
- The dot product equals to the result of addition of all the multiplied components
- $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3.$
- Example:
  - $\begin{bmatrix} 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 - 1 = 1$
  - $\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ \sqrt{2} \end{bmatrix} = 3 \times 2 + (-1) \times 1 + 0 \times \sqrt{2} = \ 5$

How to dot two vectors

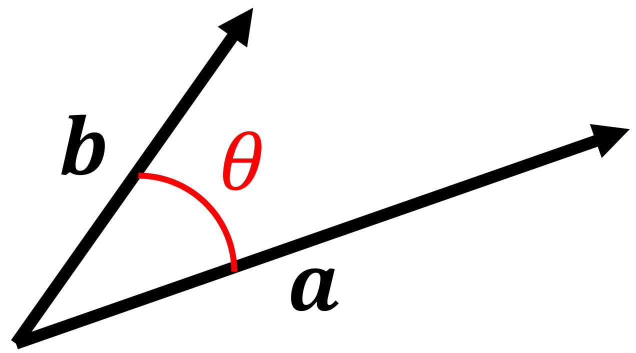
- Dot product can be computed only between vectors of same dimension.
- Dot product is commutative
  - $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
  - $\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ \sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} = 3\times 2 + (-1)\times 1 + 0\times\sqrt{2} = 5$

Geometric interpretation of a dot product

- The result of a dot product of vectors is a scalar, and cannot be depicted as a vector.
- However, this scalar value is proportional to the cosine of the angle between the vectors.
- So dot product can be computed in two different ways. One as the sum of the product of the corresponding components as mentioned earlier, and the other as

$$\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\| \, \|\boldsymbol{b}\| \, \cos\theta$$

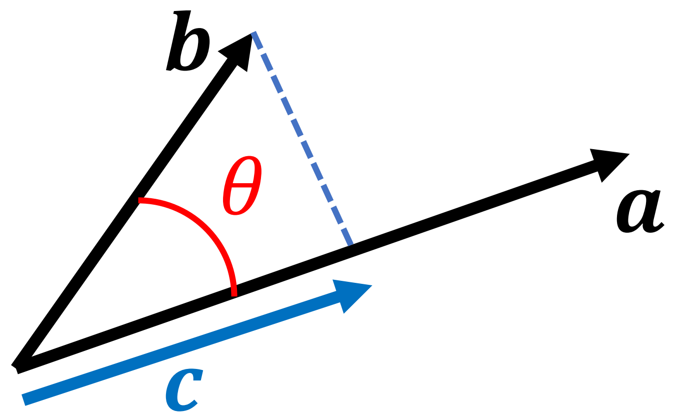
- Both computation methods will yield the same result.
- Example:
  - $\boldsymbol{a} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$
  - $\boldsymbol{a} \cdot \boldsymbol{b} = 2 \times 2 \times \cos 60^{\circ} = 2$
  - By earlier approach, $\boldsymbol{a} \cdot \boldsymbol{b} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ = 0+2=2



Geometric interpretation of a dot product

- $\boldsymbol{a} \cdot \boldsymbol{b} = ||\boldsymbol{a}||\ ||\boldsymbol{b}||\ \cos\theta$
- What if $\boldsymbol{a}$ is a unit vector ($||\boldsymbol{a}||=1$)
  - $\boldsymbol{a} \cdot \boldsymbol{b}$ would be the length of the perpendicular projection of $\boldsymbol{b}$ on $\boldsymbol{a}$

- Vector $\boldsymbol{c}$ is the image of $\boldsymbol{b}$ on $\boldsymbol{a}$
- Direction of $\boldsymbol{c}$ is the same as $\boldsymbol{a}$
- Magnitude of $\boldsymbol{c}$ is
  - $||\boldsymbol{c}|| = ||\boldsymbol{b}||\ \cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{||\boldsymbol{a}||}$



Dot product with itself

- From the geometrical representation of dot product it is inferred that the dot product of a vector with itself is its squared magnitude

$$\boldsymbol{a} \cdot \boldsymbol{a} = \|\boldsymbol{a}\|\|\boldsymbol{a}\| \cos 0^{\circ} = \|\boldsymbol{a}\|^2$$

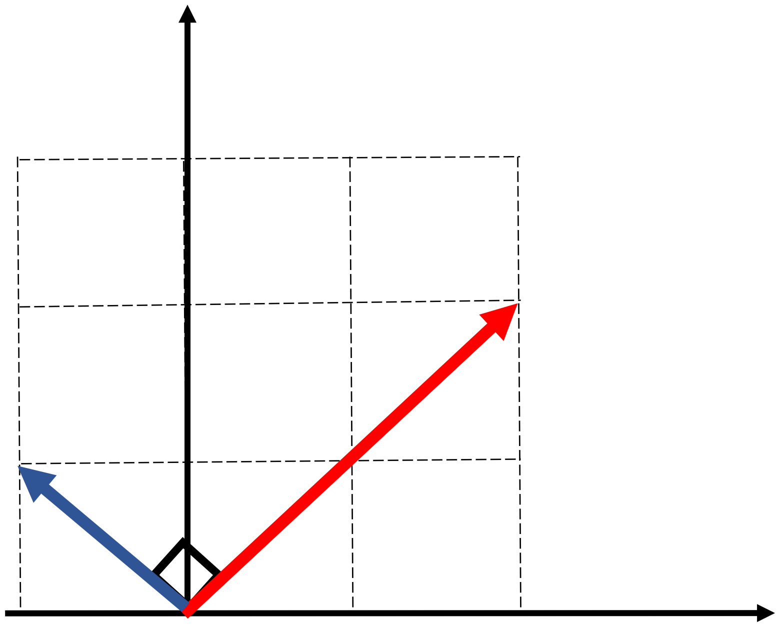
- Now if the vector is a unit vector its dot product with itself equals to 1

$$\widehat{\boldsymbol{a}} \cdot \widehat{\boldsymbol{a}} = \|\widehat{\boldsymbol{a}}\|\|\widehat{\boldsymbol{a}}\| \cos 0^{\circ} = 1$$

- Example:
  - $\boldsymbol{a} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \boldsymbol{a} \cdot \boldsymbol{a} = 2^2 + (-1)^2 = 5 = \left(\sqrt{2^2 + (-1)^2}\right)^2$
  - $\widehat{\boldsymbol{a}} = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}, \widehat{\boldsymbol{a}} \cdot \widehat{\boldsymbol{a}} = 0.6^2 + 0.8^2 = 1 = 1 \times 1$

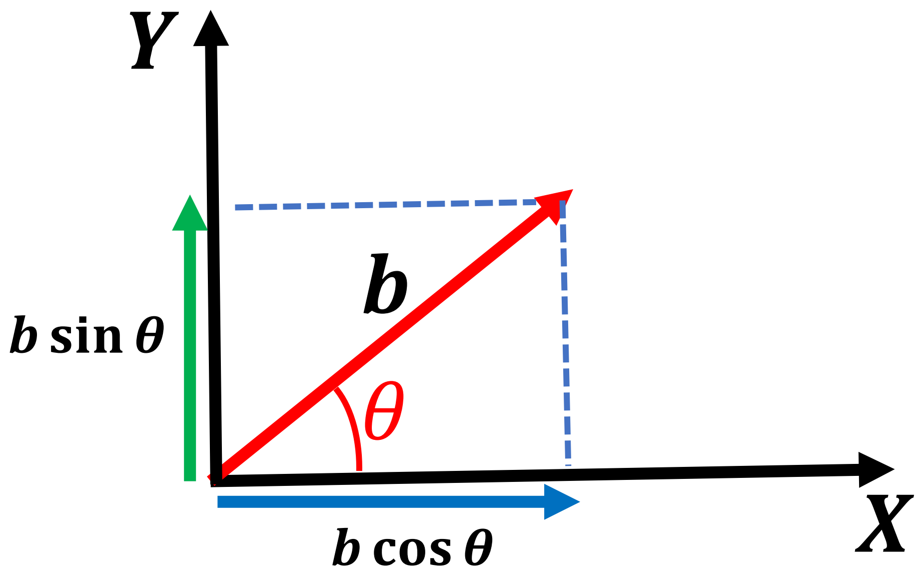
Dot product of perpendicular vectors

- From the geometric representation of dot product, it is inferred that the dot product of two perpendicular vector is equal to zero, since $\cos 90^{\circ} = 0$.
- Example:
  - $\boldsymbol{a} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
  - $\boldsymbol{a} \cdot \boldsymbol{b} = 2 \times (-1) + 2 \times 1 = 0$
  - $\boldsymbol{a} \cdot \boldsymbol{b} = \sqrt{8} \times \sqrt{2} \times \cos 90^{\circ} = 0$



Application of Dot product

- Decomposition of a vector into its orthogonal components.
- b = $b \cos\theta \, \hat{X} + b \sin\theta \, \hat{Y}$



What is the cross product

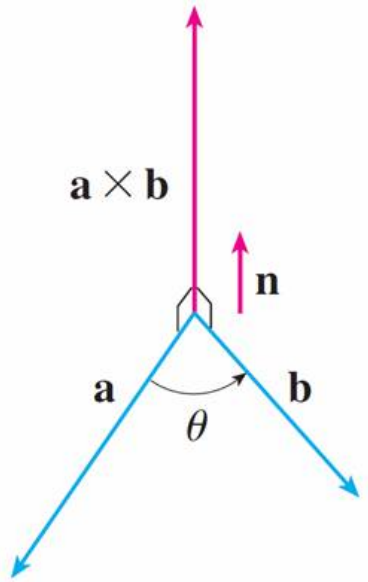
- Cross product is computed between (N-1) vectors in a N-D space, where $N \geq 3$
- The result of cross product is a **vector** in N-D space. Hence it is also called *vector product.*
- The cross product vector is perpendicular to all the (N-1) vectors used to compute it.
- Let, a = $[a_1, a_2, a_3]$ & b = $[b_1, b_2, b_3]$ then,

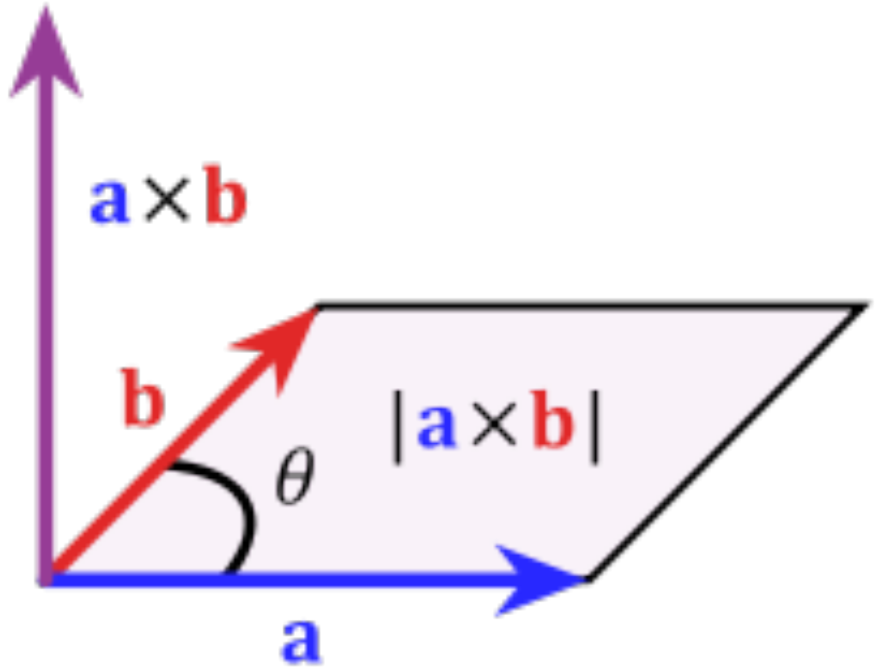
$$a \times b = [\, a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1]$$

(we will discuss about this again when we study determinants.)

Geometric interpretation of cross product

- a x b $= ||\boldsymbol{a}||\ ||\boldsymbol{b}||\ \sin\theta\hat{n}$, where θ is the angle between the two vectors and $\hat{n}$ is a unit vector representing the direction of the resultant vector.
  - Direction of a x b is perpendicular to both vectors a and b following the right hand rule.
  - |a x b| represents the area of the parallelogram determined by these vectors as adjacent sides.





Cross product (using Determinants)

EXAMPLE 1 Compute $\mathbf{u}\times\mathbf{v}$ and $\mathbf{v}\times\mathbf{u}$ for $\mathbf{u}=\langle 2,3,5\rangle$ and $\mathbf{v}=\langle 6,7,9\rangle$.

Solution: To do so, we construct the vector of determinants in (2),

$$\mathbf{u}\times\mathbf{v}=\left\langle \begin{vmatrix} 3 & 5 \\ 7 & 9 \end{vmatrix}, \begin{vmatrix} 5 & 2 \\ 9 & 6 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ 6 & 7 \end{vmatrix} \right\rangle$$

and then we use (1) to evaluate the determinants:

$$\mathbf{u}\times\mathbf{v}=\langle 3\cdot 9-7\cdot 5, 5\cdot 6-9\cdot 2, 2\cdot 7-6\cdot 3\rangle=\langle -8,12,-4\rangle \qquad (3)$$

Notice however that $\mathbf{v}\times\mathbf{u}$ is

$$\mathbf{v}\times\mathbf{u}=\left\langle \begin{vmatrix} 7 & 9 \\ 3 & 5 \end{vmatrix}, \begin{vmatrix} 9 & 6 \\ 5 & 2 \end{vmatrix}, \begin{vmatrix} 6 & 7 \\ 2 & 3 \end{vmatrix} \right\rangle=\langle 8,-12,4\rangle$$

That is, $\mathbf{v}\times\mathbf{u}=-(\mathbf{u}\times\mathbf{v})$, which can be shown to be true in general. Indeed, each of the following follow from direct calculation.

Cross product (using Determinants)

- Taking the dot product of u x v with either u or v results in a zero vector. This proves that u x v is orthogonal to both u and v.

$$[-8, 12, -4] \ \cdot [2, 3, 5] = [0, 0, 0]$$

$$[-8, 12, -4] \ \cdot [6, 7, 9] = [0, 0, 0]$$

Cross product of parallel vectors

- From the geometrical representation of cross product it is inferred that the cross product of parallel vectors is a zero vector

$$\text{a x b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \sin 0^{\circ} = 0$$

- i.e. cross product of a vector with itself is zero vector

$$\text{a x a} = \|\boldsymbol{a}\|\|\boldsymbol{a}\| \sin 0^{\circ} = 0$$

Cross product of orthogonal vectors

- From the geometric representation of cross product, it is inferred that the cross product of two orthogonal vector is the product of their magnitude.

$$\text{a x b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \sin 90^{\circ} = \|\boldsymbol{a}\|\|\boldsymbol{b}\|$$

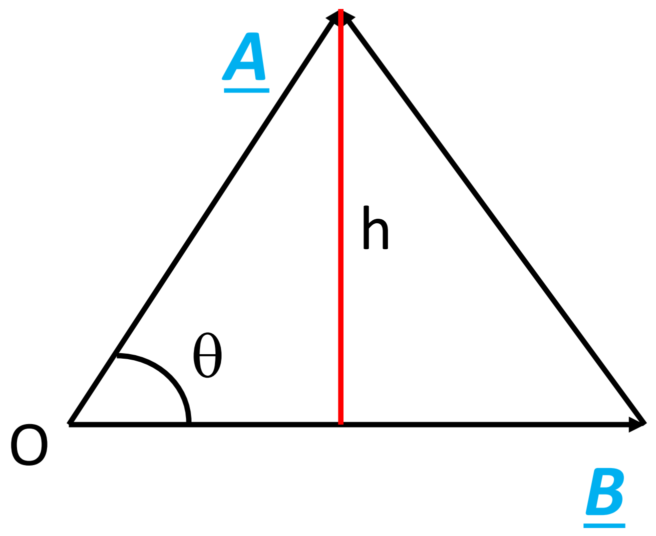
Application of Cross product

- To find the area of a triangle.

$$\text{Area of triangle} = \frac{1}{2}\, h \; |\underline{\boldsymbol{B}}|$$

$$= \frac{1}{2}\, |\underline{\boldsymbol{A}}| \sin\theta \; |\underline{\boldsymbol{B}}|$$

$$= \frac{1}{2}\, |\, \underline{\boldsymbol{A}} \times \underline{\boldsymbol{B}} \,|$$

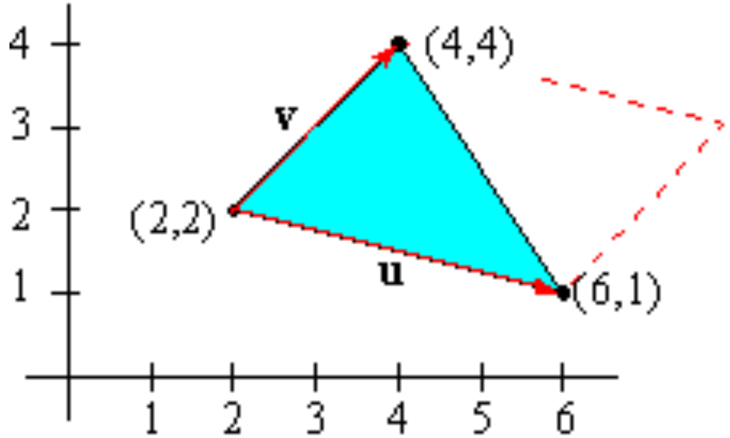


Application of Cross product

- Example of Area of Triangle

Find the area of triangle with vertices

P1 (2,2), P2(4,4) and P3(6,1) :



Solution: It is easy to see that $\mathbf{u} = \langle 2, 2 \rangle$ and $\mathbf{v} = \langle 4, -1 \rangle$. As vectors in $\mathbb{R}^3$, we have $\mathbf{u} = \langle 2, 2, 0 \rangle$ and $\mathbf{v} = \langle 4, -1, 0 \rangle$. Thus, their cross product is

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \left\langle \begin{vmatrix} 2 & 0 \\ 4 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix}, \begin{vmatrix} 2 & 2 \\ 4 & -1 \end{vmatrix} \right\rangle \\ &= \langle 0, 0,\ 2 \cdot (-1) - 4 \cdot 2 \ \rangle \\ &= \langle 0, 0, -10 \rangle \end{aligned}$$

Since the triangle has half of the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$, the area of the triangle is

$$Area = \|\mathbf{u} \times \mathbf{v}\| = \frac{1}{2}\sqrt{0^2 + 0^2 + (-10)^2} = 5\ units^2$$

Note the area of the parallelogram formed by u and v will be 2 * 5 units^2 = 10units^2

Algebraic Properties of dot and cross products

11 **Theorem** If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

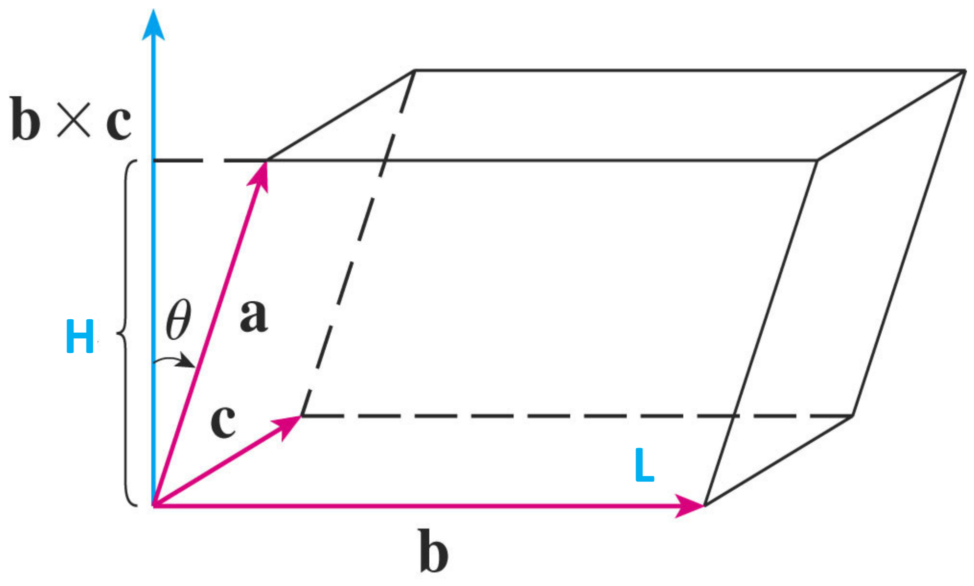
Dot Product vs Cross Product

Dot product	Cross product
Result of a dot product is a scalar quantity.	Result of a cross product is a vector quantity.
It follows commutative law.	It doesn’t follow commutative law.
Dot product of vectors in the same direction is maximum.	Cross product of vectors in same direction is zero.
Dot product of orthogonal vectors is zero.	Cross product of orthogonal vectors is maximum.
It doesn’t follow right hand system.	It follows right hand system.
It is used to find projection of vectors.	It is used to find a third vector.
It is represented by a dot (.)	It is represented by a cross (x)

Scalar Triple Product

- Scalar triple product of vectors a, b, c is referred to as $a \cdot (b \; x \; c)$
- Geometrically it represents the volume of a parallelopiped

Volume of the parallelepiped



= (height **H**) (area of the parallelogram **L**)

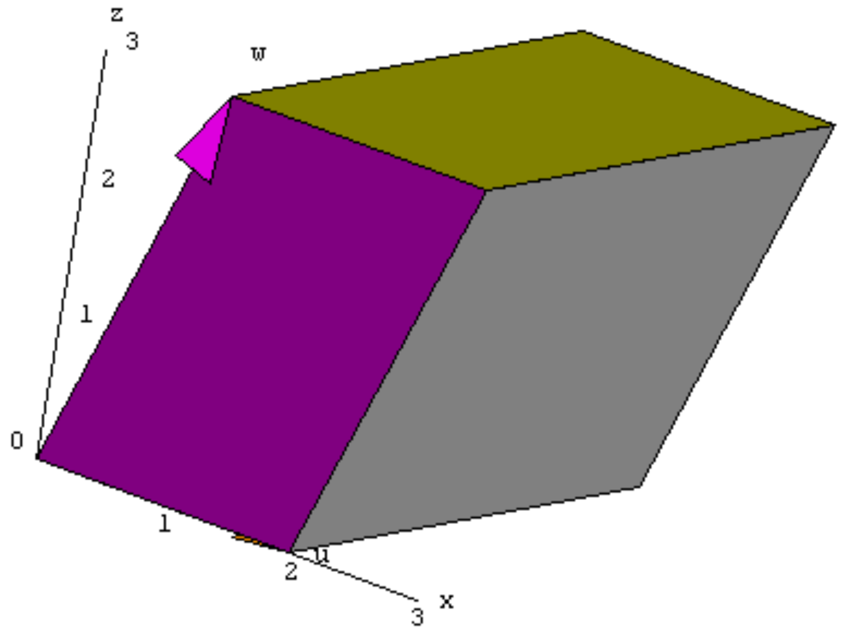
= $(|\boldsymbol{a} \cos \boldsymbol{\theta}|) (| \boldsymbol{b} \times \boldsymbol{c} |)$

= $|\boldsymbol{a}| \, (| \boldsymbol{b} \times \boldsymbol{c} |) \, |\cos \boldsymbol{\theta}|$

= $|\boldsymbol{a} \, . \, (\boldsymbol{b} \times \boldsymbol{c}) \, |$

Scalar Triple Product

EXAMPLE 7 Find the volume of the parallelpiped spanned by $\mathbf{u} = \langle 2, 0, 0 \rangle$, $\mathbf{v} = \langle 1, 3, 0 \rangle$, and $\mathbf{w} = \langle 1, 0, 3 \rangle$. The figure below is drawn as if all vectors have their initial points at the origin.



Solution: The cross product of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \times \mathbf{v} = \left\langle \begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} \right\rangle = \langle 0, 0, 6 \rangle$$

The dot product with $\mathbf{w}$ yields the volume:

$$Volume = |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = |\langle 1, 0, 3 \rangle \cdot \langle 0, 0, 6 \rangle| = 18$$