Lecture 3: Matrix and Matrix Operations

- Representation, row vector, column vector, element of a matrix.
- Examples of matrix representations – Tables and spreadsheets
- Scalar-Matrix operation: Scaling a matrix
- Vector-Matrix operation: Multiplication of matrix and a vector.
  - Computation as dot product between vectors
  - Interpreting as linear combination of column vectors
  - Properties of matrix-vector multiplication
- Matrix-Matrix operations
  - Multiplication
    - Example: Feedback control systems
    - Example: Transitive closure
  - Addition of matrices
  - Transpose of a matrix
- Special Matrices
  - Square Matrix
  - Identity Matrix
  - Diagonal Matrix
  - Symmetric Matrix
  - Anti-symmetric Matrix
  - Upper/Lower Triangular Matrices
  - Orthogonal Matrix

Matrix Representation

- Matrices can be assumed as a sequence of vectors of the same dimension.
- Representation is similar to a 2D array in programming.
- Example:

$$\boldsymbol{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \boldsymbol{a}_2 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}, \boldsymbol{a}_3 = \begin{bmatrix} 0.5 \\ 5 \\ -1 \end{bmatrix}, \boldsymbol{b}_1 = \begin{bmatrix} 2 & 1 \end{bmatrix}, \boldsymbol{b}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

  - $A = [\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3] = \begin{bmatrix} 2 & -1 & 0.5 \\ -1 & 1 & 5 \\ 0 & -2 & -1 \end{bmatrix}$ (using Column vectors)
  - $B = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ (using Row vectors)

Size of a Matrix

- Size of a matrix is represented as (# of rows)×(# of columns)
- Example:
  - $A = \begin{bmatrix} 2 & -1 & 0.5 \\ -1 & 1 & 5 \\ 0 & -2 & -1 \end{bmatrix}$ is a 3×3 matrix
  - $B = \begin{bmatrix} 2 & \sqrt{2} \\ -1 & 1 \\ 0 & 0 \end{bmatrix}$ is a 3 × 2 matrix
- $A_{i,j}$ is the component of A in the i-th row and j-th column.
  - In above example $A_{3,2} = -2$

Matrix representation of vectors

- Vectors can also be considered as matrices.
- Row vectors are $1 \times n$ matrices, where n is the number of components.
- Column vectors are $n \times 1$ matrices, where n is the number of components.
- Example:
  - Column vector $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ is a 3×1 matrix.
  - Row vector $\begin{bmatrix} 2 & 1 \end{bmatrix}$ is 1×2 matrix.

When are two matrices equal?

- Two matrices are equal if their sizes and corresponding elements are equal.

- Example:
  - $\begin{bmatrix} 2 & -1 & 0.5 \\ -1 & 1 & 5 \\ 0 & -2 & -1 \end{bmatrix} = \begin{bmatrix} \frac{4}{2} & -1 & \frac{1}{2} \\ -1 & 1 & 5 \\ 1-1 & -2 & -1 \end{bmatrix}$, both are 3×3 matrices.

Scalar - Matrix Operations

- Scalars can be multiplied with matrices.
- Similar to scalar-vector multiplication, the scalar is multiplied with every element of the matrix
- Example:
  - $2\begin{bmatrix} 2 & 3 \\ 2 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 4 & 2 \\ 0 & -2 \end{bmatrix}$
  - $-1\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$

Vector and Matrix Operations

A vector can be multiplied with a matrix in two possible ways. i.e. Matrix-Vector multiplication or Vector-Matrix multiplication.

Matrix-Vector Multiplication: $Ab = c$

- Matrix $\boldsymbol{A}_{m\times n}$ can be multiplied with column vector $\boldsymbol{b}_{n\times 1}$ to get a column vector $\boldsymbol{c}_{m\times 1}$.

Vector-Matrix Multiplication: $aB = c$

- Row vector $\boldsymbol{a}_{1\times n}$ can be multiplied with matrix $B_{n\times m}$ to get a row vector $\boldsymbol{c}_{1\times m}$.

Matrix-Vector Multiplication

- Matrix $A_{m\times n}$ can be multiplied with column vector $\boldsymbol{b}_{n\times 1}$ to get a column vector $\boldsymbol{c}_{m\times 1}$.
- i-th component of $\boldsymbol{c}$ is dot product of i-th row of A with $\boldsymbol{b}$.

$$A\boldsymbol{b} = \begin{array}{r} \boldsymbol{a_1} \rightarrow \\ \boldsymbol{a_2} \rightarrow \end{array} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{a_1} \cdot \boldsymbol{b} \\ \boldsymbol{a_2} \cdot \boldsymbol{b} \end{bmatrix}$$

Matrix-Vector Multiplication

- Number of columns of the matrix and the dimension (number of components) of the column vector must be equal.

$$\begin{bmatrix} 2 & 3 \\ 2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

- What if the column vector is orthogonal to the rows of the matrix?
  - Obviously, the result is a zero vector, since the dot product of orthogonal vectors is zero.

$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Vector-Matrix Multiplication

- Row vector $\boldsymbol{a}_{1\times n}$ can be multiplied with matrix $B_{n\times m}$ to get a row vector $\boldsymbol{c}_{1\times m}$.
- i-th component of $\boldsymbol{c}$ is dot product of $\boldsymbol{a}$ and the i-th column of B.

$$\boldsymbol{a}B = [a_1 \quad a_2 \quad a_3]\begin{array}{c} \begin{array}{cc} \boldsymbol{b}_1 & \boldsymbol{b}_2 \\ \downarrow & \downarrow \end{array} \\ \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \end{array} = [\boldsymbol{a}\cdot\boldsymbol{b}_1 \quad \boldsymbol{a}\cdot\boldsymbol{b}_2]$$

Vector-Matrix Multiplication

- Number of components of the row vector and number of rows of the matrix must be equal.

$$[1 \quad 0]\begin{bmatrix} 2 & 2 & 0 \\ 3 & 1 & -1 \end{bmatrix} = [2 \quad 2 \quad 0]$$

- What if the row vector is orthogonal to columns of matrix?
  - The result is a zero vector.

$$[1 \quad -2]\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} = [0 \quad 0]$$

Matrix - Matrix Operations

- Matrix-Matrix Multiplication

$$\boldsymbol{AB}$$

- Matrix-Matrix Addition

$$\boldsymbol{A}+\boldsymbol{B}$$

- Matrix-Matrix Subtraction

$$\boldsymbol{A}-\boldsymbol{B}$$

Matrix-Matrix Multiplication

- Two matrices $\boldsymbol{A}_{n\times m}$ and $\boldsymbol{B}_{k\times l}$ can be multiplied as $\boldsymbol{AB}$, only if $m = k$
- The result is a matrix of the size of $n \times l$ ($\boldsymbol{C}_{n\times l}$)

- Example:
  - $\begin{bmatrix} 2 & -1 & 2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$ is possible

  - $\begin{bmatrix} 2 & -1 & 2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is NOT possible

Matrix-Matrix Multiplication

- Multiplication of matrices $\boldsymbol{AB}$ is nothing but the dot products of row vectors of $\boldsymbol{A}$ with the column vectors of $\boldsymbol{B}$
- The component (i, j) of the resulting matrix is the result of the dot product of i-th row vector of $\boldsymbol{A}$ with the j-th column vector of $\boldsymbol{B}$.
- If $\boldsymbol{A}_{n\times m}$ and $\boldsymbol{B}_{m\times l}$, so $\boldsymbol{A}$ has n row vectors and $\boldsymbol{B}$ has l column vectors. Therefore, the result matrix is $n \times l$.

Matrix-Matrix Multiplication (as dot product)

$$\boldsymbol{AB} = \begin{array}{l} \boldsymbol{a}_1 \rightarrow \\ \boldsymbol{a}_2 \rightarrow \end{array} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{12} & a_{13} \end{bmatrix} \begin{array}{c} \begin{array}{cc} \boldsymbol{b}_1 & \boldsymbol{b}_2 \\ \downarrow & \downarrow \end{array} \\ \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \end{array} = \begin{bmatrix} \boldsymbol{a}_1 \cdot \boldsymbol{b}_1 & \boldsymbol{a}_1 \cdot \boldsymbol{b}_2 \\ \boldsymbol{a}_2 \cdot \boldsymbol{b}_1 & \boldsymbol{a}_2 \cdot \boldsymbol{b}_2 \end{bmatrix}$$

Matrix-Matrix Multiplication (as linear combination)

Compute AB, where $A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & 2 & 3 \end{bmatrix}$.

SOLUTION Write $B = [\,\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3\,]$, and compute:

$$A\mathbf{b}_1 = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad A\mathbf{b}_2 = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A\mathbf{b}_3 = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 \\ 1 \end{bmatrix} \qquad\qquad = \begin{bmatrix} 0 \\ 13 \end{bmatrix} \qquad\qquad = \begin{bmatrix} 21 \\ 9 \end{bmatrix}$$

Then

$$AB = A[\,\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3\,] = \begin{array}{c} \begin{bmatrix} 11 & 0 & 21 \\ 1 & 13 & 9 \end{bmatrix} \\ \uparrow \quad\;\; \uparrow \quad\;\; \uparrow \\ A\mathbf{b}_1 \;\; A\mathbf{b}_2 \;\; A\mathbf{b}_3 \end{array}$$

Each Column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

Properties of Matrix Multiplication

- $AB \neq BA$ -- not commutative
- $A(BC) = (AB)C$ -- associative law
- $A(B+C) = AB + AC$ -- left distributive law
- $(B+C)A = BA + CA$ -- right distributive law
- $r(AB) = (rA)B = A(rB)$ for any scalar r
- $IA = AI = A$
- $(AB)^T = B^T A^T$

Inner and Outer Products

- Inner Product: Inner product of two vectors results in a scalar.

$$a^T b = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = [-2 + 0 - 2] = [-4]$$

- Outer Product: Outer product of two vectors results in a matrix.

$$ab^T = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 2 \\ 0 & 0 & 0 \\ 2 & -1 & -2 \end{bmatrix}$$

Matrix Addition

- Two matrices can be added only if they have the same dimension
- In matrix addition corresponding elements of the matrices are added.
- Example:
  - $\begin{bmatrix} 2 & 3 \\ 2 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \\ 1 & 1 \end{bmatrix}$
  - $\begin{bmatrix} 1 & 0.5 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ Cannot be added

Properties of Matrix Addition

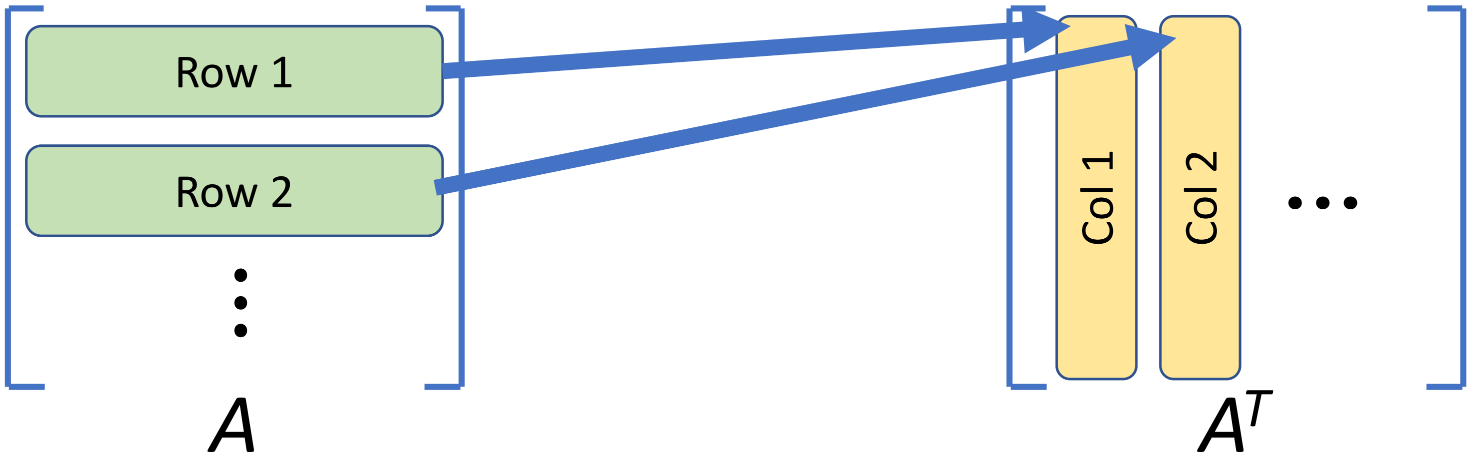
- $A + B = B + A$ -- commutative law
- $A + (B + C) = (A + B) + C$ -- associative law
- $A + 0 = 0 + A = A$
- $A – A = 0$
- $(A + B)^T = A^T + B^T$

Matrix Subtraction

- Two matrices can be subtracted only if they have the same dimension
- In matrix subtraction corresponding components of matrices are subtracted.
- Example:
  - $\begin{bmatrix} 2 & 3 \\ 2 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 0 \\ -1 & -3 \end{bmatrix}$
  - $\begin{bmatrix} 1 & 0.5 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ Cannot be subtracted

Matrix Transpose

- Transpose of a matrix is a matrix where its rows are columns of the original matrix. (And its columns are the rows of the original matrix.)



- Transpose converts row vectors to column vectors and vice-versa

Matrix Transpose

- Transpose of matrix A is noted as A^T (NOT A to the power of T).
- If size of A is $m \times n$,then size of A^T is $n \times m$.
- Component (i, j) in original matrix is the component (j, i) in the transpose matrix.
- Example:
  - $A = \begin{bmatrix} 2 & \sqrt{2} \\ -1 & 1 \\ 0 & 0 \end{bmatrix}, A^T = \begin{bmatrix} 2 & -1 & 0 \\ \sqrt{2} & 1 & 0 \end{bmatrix}$

Properties of Matrix Transpose

- Transpose of transpose of a matrix is the matrix itself
- $(A^T)^T = A$
- Superposition property
- $(A + B)^T = A^T + B^T$
- Scaling property
- $(cA)^T = cA^T$
- Transpose of a product of matrices equals the product of their transpose in the reverse order
- $(AB)^T = B^T A^T$

Special Matrix

- Square Matrix
- Symmetric Matrix
- Anti-Symmetric Matrix
- Diagonal Matrix
- Identity Matrix
- Upper/Lower Triangular Matrix
- Orthogonal Matrix

Square matrix

- Square matrix is a matrix whose number of rows is equal to its number of columns.
- Square matrix A is a $n \times n$ matrix.
- Example:
  - $\begin{bmatrix} 2 & -1 & 0.5 \\ -1 & 1 & 5 \\ 0 & -2 & -1 \end{bmatrix}$ is a square 3×3 matrix.
  - $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ is a square 2×2 matrix.

Symmetric matrix

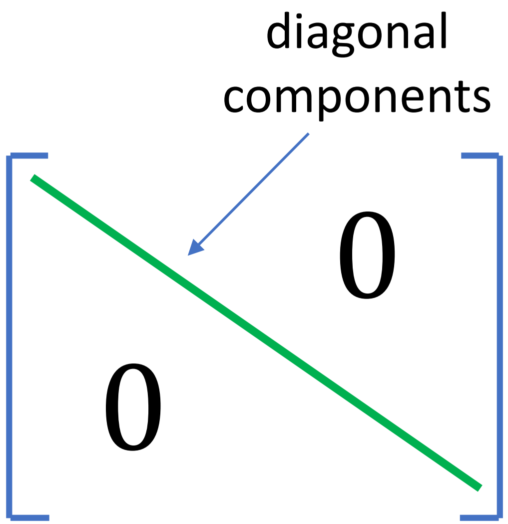
- A symmetric matrix is a square matrix which is equal to its transpose.
- $A = A^T$ implies A is a symmetric matrix.
- Components (i, j) and (j, i) of the symmetric matrix are equal. $(A_{i,j} = A_{j,i})$
- Example:
  - $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$
  - $\begin{bmatrix} 2 & -1 & 0.5 \\ -1 & 1 & 5 \\ 0.5 & 5 & -1 \end{bmatrix}$

Antisymmetric matrix

- An antisymmetric matrix is a square matrix which is equal to its transpose multiplied by -1.
- $A = -A^T$ if A is an antisymmetric matrix.
- Components (i, j) and (j, i) of an antisymmetric matrix are equal in value and different in sign. ($A_{i,j} = -A_{j,i}$)
- Example:
  - $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
  - $\begin{bmatrix} 0 & -1 & 0.5 \\ 1 & 0 & -5 \\ -0.5 & 5 & 0 \end{bmatrix}$

Diagonal matrix

- A diagonal matrix is a square matrix whose non-diagonal components are all zero.
- Diagonal matrices are symmetric, as well.
- Example:
  - $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$
  - $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$



Identity matrix

- Identity matrix is a diagonal matrix whose diagonal values are all 1.
- Identity matrix is denoted by I.
- Example:
  - $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a 2×2 identity matrix
  - $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a 3×3 identity matrix.

Upper/Lower triangular matrix

- Upper triangular matrix is a square matrix whose elements below the diagonal are all 0.
- Lower triangular matrix is a square matrix whose elements above the diagonal are all 0.
- Example:
  - $\begin{bmatrix} 2 & -1 & 0.5 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix}$ is an upper triangular matrix
  - $\begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 0.5 & 5 & -1 \end{bmatrix}$ is a lower triangular matrix

Orthogonal/Orthonormal Matrix

- Matrix A is an orthonormal matrix if it is a square matrix and
- $AA^T = A^TA = I$, where I is an identity matrix.

- Every row is orthogonal to every other row, and every column is orthogonal to every other column. Every row and every column is a unit vector.

$$AA^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1 \cdot \boldsymbol{a}_1 & \boldsymbol{a}_1 \cdot \boldsymbol{a}_2 & \boldsymbol{a}_1 \cdot \boldsymbol{a}_3 \\ \boldsymbol{a}_2 \cdot \boldsymbol{a}_1 & \boldsymbol{a}_2 \cdot \boldsymbol{a}_2 & \boldsymbol{a}_2 \cdot \boldsymbol{a}_3 \\ \boldsymbol{a}_3 \cdot \boldsymbol{a}_1 & \boldsymbol{a}_3 \cdot \boldsymbol{a}_2 & \boldsymbol{a}_3 \cdot \boldsymbol{a}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Orthogonal/Orthonormal Matrix

- Example:

$$\begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(Orthogonal matrices are a very important class of matrices. They have fascinating properties which we will see later.)