Solving Consistent Linear Systems
Matrix Notation

• An **augmented matrix** of a system consists of the coefficient matrix with an added column containing the constants from the right sides of the equations.

• For the given system of equations,

\[
\begin{bmatrix}
1 & -2 & 1 & 0 \\
0 & 2 & -8 & 8 \\
-4 & 5 & 9 & -9
\end{bmatrix}
\]

is called the **augmented matrix** of the system.
Augmented matrix

• Augmented matrix is a matrix obtained by appending the columns of two given matrices.

• In the case of $Ax = b$, we are interested to work with the augmented matrix $[A \mid b]$

• Example:
  • If $A = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$ then $[A \mid b]$ is $\begin{bmatrix} 1 & -2 & 1 \\ 3 & 2 & 11 \end{bmatrix}$

  • Note $\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 11 \\ 2 \end{bmatrix}$ cannot be augmented, since their number of rows are different.
Elimination

- Elimination is a systematic way to solve linear equations.
- Example:

\[
\begin{align*}
\begin{cases}
    x_1 - 2x_2 &= 1 \\
    3x_1 + 2x_2 &= 11
\end{cases}
\end{align*}
\]

We can remove the factor of \( x_1 \) in the second equation by subtracting three times of the first equation from the second equation.

\[
3x_1 + 2x_2 - 3(x_1 - 2x_2) = 11 - 3(1) \implies 8x_2 = 8
\]

As shown above, the second equation becomes \( 8x_2 = 8 \), and independent of \( x_1 \). Therefore, we are left with:

\[
\begin{align*}
\begin{cases}
    x_1 - 2x_2 &= 1 \\
    8x_2 &= 8
\end{cases}
\end{align*}
\]

From the new second equation we get \( x_2 = 1 \).
- By back-substituting the value of \( x_2 = 1 \) in the first equation, it becomes \( x_1 - 2 = 1 \).
- So, we can find \( x_1 = 3 \) as well.
Elimination

• Elimination can be done on augmented matrix, to help solving matrix equations. The elimination process is also called row reduction.

• Example:
  - \[
  \begin{bmatrix}
  1 & -2 \\
  3 & 2 \\
  \end{bmatrix}
  \begin{bmatrix}
  x \\
  y \\
  \end{bmatrix} =
  \begin{bmatrix}
  1 \\
  11 \\
  \end{bmatrix}
  \]
  - Construct the augmented matrix: \[
  \begin{bmatrix}
  1 & -2 & 1 \\
  3 & 2 & 11 \\
  \end{bmatrix}
  \]

  • Subtract three times of first row from the second row.
    - \[
    \begin{bmatrix}
    1 & -2 & 1 \\
    3 - 3(1) & 2 - 3(-2) & 11 - 3(1) \\
    \end{bmatrix} =
    \begin{bmatrix}
    1 & -2 & 1 \\
    0 & 8 & 8 \\
    \end{bmatrix}
    \]

  • Now, if we convert the augmented matrix back to matrix equation form we have:
    - \[
    \begin{bmatrix}
    1 & -2 \\
    0 & 8 \\
    \end{bmatrix}
    \begin{bmatrix}
    x \\
    y \\
    \end{bmatrix} =
    \begin{bmatrix}
    1 \\
    8 \\
    \end{bmatrix}
    \]

  • \( A \) becomes an upper-triangular matrix. We can easily find \( y \) and then \( x \), by back-substitution.
Row operations

- Within the process of elimination, the following operations are valid:
  1. One row is replaced by a linear combination of that row with another row.

\[
\begin{bmatrix}
a & b \\
c & d \\
\end{bmatrix} \rightarrow \begin{bmatrix}
a + \gamma a & b + \gamma b \\
c + \gamma a & d + \gamma b \\
\end{bmatrix}, \gamma \in \mathbb{R}
\]

2. A row is replaced by a scaled version of the same row.

\[
\begin{bmatrix}
a & b \\
c & d \\
\end{bmatrix} \rightarrow \begin{bmatrix}
a & b \\
\gamma c & \gamma d \\
\end{bmatrix}, \gamma \in \mathbb{R}
\]

3. Two rows are exchanged.

\[
\begin{bmatrix}
a & b \\
c & d \\
\end{bmatrix} \rightarrow \begin{bmatrix}
c & d \\
a & b \\
\end{bmatrix}
\]
Row operations

• The goal is to do the row reduction on the augmented matrix to reach to upper-triangular matrix, and solve the equation by back-substitution

1. Order the equations such that the main diagonal coefficients are non-zero.

2. -Use the first equation to eliminate the first variable from all the equations (rows) bellow it.
   - Use the second equation to eliminate the second variable from all the equations (rows) bellow.
   - And so on

3. If any diagonal element become zero, swap that equation (row) with any other equation below it.
Elimination procedure

• Example: Do the row reduction on the following linear equations to reach to upper-triangular matrix, and solve the equation by back-substitution.

\[
\begin{align*}
  x_1 - 2x_2 + x_3 &= 0 \\
  2x_2 - 8x_3 &= 8 \\
  -4x_1 + 5x_2 + 9x_3 &= -9
\end{align*}
\]

• The equivalent matrix equation is:

\[
\begin{bmatrix}
  1 & -2 & 1 & 0 \\
  0 & 2 & -8 & 8 \\
  -4 & 5 & 9 & -9
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix}
= 
\begin{bmatrix}
  0 \\
  8 \\
  -9
\end{bmatrix}
\]

• The augmented matrix becomes:

\[
\begin{bmatrix}
  1 & -2 & 1 & 0 \\
  0 & 2 & -8 & 8 \\
  -4 & 5 & 9 & -9
\end{bmatrix}
\]
Elimination procedure

• Example Cont’d
  • The first element in the second row is already 0.
  • If we replace 4-times of the first row plus the third row, the first element of the third row becomes 0. \((r_3 \leftarrow 4r_1 + r_3)\)
  \[
  \begin{bmatrix}
  1 & -2 & 1 & 0 \\
  0 & 2 & -8 & 8 \\
  -4 + 4(1) & 5 + 4(-2) & 9 + 4(1) & -9 + 4(0)
  \end{bmatrix}
  =
  \begin{bmatrix}
  1 & -2 & 1 & 0 \\
  0 & 2 & -8 & 8 \\
  0 & -3 & 13 & -9
  \end{bmatrix}
  \]
  • Now all the elements below the first diagonal element become 0.
  • We continue elimination in the second column by adding \(\frac{3}{2}\) of the second to the third row. \((r_3 \leftarrow \frac{3}{2}r_2 + r_3)\)
  \[
  \begin{bmatrix}
  1 & -2 & 1 & 0 \\
  0 & 2 & -8 & 8 \\
  0 & -3 + \frac{3}{2}(2) & 13 + \frac{3}{2}(-8) & -9 + \frac{3}{2}(8)
  \end{bmatrix}
  =
  \begin{bmatrix}
  1 & -2 & 1 & 0 \\
  0 & 2 & -8 & 8 \\
  0 & 0 & 1 & 3
  \end{bmatrix}
  \]
  • The final result is called the Echelon form of the matrix (definition later)
Echelon vs. reduced echelon form

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.
Echelon vs. reduced echelon form

• The following matrices are in echelon form:
  \[
  \begin{bmatrix}
    1 & -2 & 1 & 0 \\
    0 & 2 & -8 & 8 \\
    0 & 0 & 1 & 3
  \end{bmatrix},
  \begin{bmatrix}
    1 & -2 & 1 & 0 \\
    0 & 2 & -8 & 8 \\
    0 & 0 & 0 & 0
  \end{bmatrix}
  \]

• The following matrices are in reduced echelon form
  \[
  \begin{bmatrix}
    1 & 0 & 0 & 29 \\
    0 & 1 & 0 & 16 \\
    0 & 0 & 1 & 3
  \end{bmatrix},
  \begin{bmatrix}
    1 & 0 & 1 & 15 \\
    0 & 1 & 2 & 16 \\
    0 & 0 & 0 & 0
  \end{bmatrix}
  \]
Elimination procedure Contd.

• Continuing with the earlier example: We can compute the solution from Echelon form using back substitution, or continue with the row reduction to create what is called the reduced echelon form.

• Solving using back substitution
  • Converting back the augmented matrix to equation form we have:
    \[
    \begin{align*}
    x_1 - 2x_2 + x_3 &= 0 \\
    2x_2 - 8x_3 &= 8 \\
    x_3 &= 3
    \end{align*}
    \]
  • We have \( x_3 = 3 \).
  • Back-substitute \( x_3 = 3 \) in the second equation \( 2x_2 - 24 = 8 \), so \( x_2 = 16 \).
  • Back-substitute \( x_3 = 3 \) and \( x_2 = 16 \) in the first equation \( x_1 - 32 + 3 = 0 \), so \( x_1 = 29 \).
  • We found the matrix equation has one solution which is \[ \begin{bmatrix} 29 \\ 16 \\ 3 \end{bmatrix} \].
Elimination procedure

• Solving using reduced echelon form from the last example. \[
\begin{bmatrix}
1 & -2 & 1 & 0 \\
0 & 2 & -8 & 8 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

• We should make the all elements above and below the leading non-zero element (aka pivot) in each row zero.

• Add 8-times of the third row to the second row and replace the second row with the result. \((r_2 \leftarrow 8r_3 + r_2)\)

\[
\begin{bmatrix}
1 & -2 & 1 & 0 \\
0 & 2 & -8 + 8(1) & 8 + 8(3) \\
0 & 0 & 1 & 3
\end{bmatrix} = \begin{bmatrix}
1 & -2 & 1 & 0 \\
0 & 2 & 0 & 32 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

• Now subtract the third row from the first row and replace the first row with the result. \((r_1 \leftarrow r_3 - r_1)\)

\[
\begin{bmatrix}
1 & -2 & 1 - 1(1) & 0 - 1(3) \\
0 & 2 & 0 & 32 \\
0 & 0 & 1 & 3
\end{bmatrix} = \begin{bmatrix}
1 & -2 & 0 & -3 \\
0 & 2 & 0 & 32 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]
Elimination procedure

• Example Cont’d
  • We are done with the third column, so we should make the element above the diagonal element in the second column become 0.
  • Add the second row with the first row and replace the first row with result. \((r_1 \leftarrow r_2 + r_1)\)
    \[
    \begin{bmatrix}
    1 & -2 + 1(2) & 0 & -3 + 1(32) \\
    0 & 2 & 0 & 32 \\
    0 & 0 & 1 & 3
    \end{bmatrix}
    =
    \begin{bmatrix}
    1 & 0 & 0 & 29 \\
    0 & 2 & 0 & 32 \\
    0 & 0 & 1 & 3
    \end{bmatrix}
    \]
  • Divide the second row by 2, in order to make diagonals all 1.
    \[
    \begin{bmatrix}
    1 & 0 & 0 & 29 \\
    0 & 2 & 0 & 32 \\
    0 & 1 & 3
    \end{bmatrix}
    =
    \begin{bmatrix}
    1 & 0 & 0 & 29 \\
    0 & 1 & 0 & 16 \\
    0 & 0 & 1 & 3
    \end{bmatrix}
    \]
  • We have found the reduced echelon form of the augmented matrix, and it directly gives the solution that we found earlier.
Pivots of a matrix

• First non-zero element in each row of the echelon form of $A$ is a pivot.
• Note that in augmented matrices pivots must be belong to $A$ and NOT $b$.
• Each column of $A$ has at most one pivot.
• The columns of $A$ that has a pivot are called pivot columns.

• Example:
  - $A = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

  - The boxed elements are pivots.
Rank of matrix

- Number of pivots in a matrix is the rank of that matrix.
- Example:

\[
A = \begin{bmatrix}
2 & 4 & -2 \\
0 & 1 & 1 \\
0 & 0 & 4 \\
\end{bmatrix}, \quad \text{Rank}(A) = 3, \quad B = \begin{bmatrix}
2 & 4 & -2 \\
0 & 1 & 5 \\
0 & 0 & 0 \\
\end{bmatrix}, \quad \text{Rank}(B) = 2
\]

- A square $n \times n$ matrix is full rank if rank of the matrix is $n$.
- If the rank of square matrix is not $n$ the matrix is rank deficient.
- Full rank square matrices are invertible (has inverse), where rank deficient matrices are NOT invertible.
Pivots and rank

• Example: Find the pivots and rank of the matrix in the following augmented matrix:

\[
\begin{bmatrix}
0 & -3 & -6 & 4 & 9 \\
-1 & -2 & -1 & 3 & 1 \\
-2 & -3 & 0 & 3 & -1 \\
1 & 4 & 5 & -9 & -7
\end{bmatrix}
\]

• We do the row reduction procedure until we get echelon form of the matrix.
• Since the first row begins with a 0 and we cannot make other elements of the first column zero by adding multiples of zero to them, we exchange the first row with the last row.

\[
\begin{bmatrix}
0 & -3 & -6 & 4 & 9 \\
-1 & -2 & -1 & 3 & 1 \\
-2 & -3 & 0 & 3 & -1 \\
1 & 4 & 5 & -9 & -7
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 4 & 5 & -9 & -7 \\
-1 & -2 & -1 & 3 & 1 \\
-2 & -3 & 0 & 3 & -1 \\
0 & -3 & -6 & 4 & 9
\end{bmatrix}
\]
Pivots and rank

• Example cont’d:
  • Then making elements below the pivot element of the first row 0 (in the first column).

  \[
  \begin{bmatrix}
  1 & 4 & 5 & -9 & -7 \\
  -1 & -2 & -1 & 3 & 1 \\
  -2 & -3 & 0 & 3 & -1 \\
  0 & -3 & -6 & 4 & 9
  \end{bmatrix}
  \rightarrow
  \begin{bmatrix}
  1 & 4 & 5 & -9 & -7 \\
  0 & 2 & 4 & -6 & -6 \\
  -2 & -3 & 0 & 3 & -1 \\
  0 & -3 & 6 & 4 & 9
  \end{bmatrix}
  \rightarrow
  \begin{bmatrix}
  1 & 4 & 5 & -9 & -7 \\
  0 & 2 & 4 & -6 & -6 \\
  0 & 5 & 10 & -15 & -15 \\
  0 & -3 & -6 & 4 & 9
  \end{bmatrix}
  \]

  • Now making the elements below the pivot element of the second row 0.

  \[
  \begin{bmatrix}
  1 & 4 & 5 & -9 & -7 \\
  0 & 2 & 4 & -6 & -6 \\
  0 & 5 & 10 & -15 & -15 \\
  0 & -3 & -6 & 4 & 9
  \end{bmatrix}
  \rightarrow
  \begin{bmatrix}
  1 & 4 & 5 & -9 & -7 \\
  0 & 2 & 4 & -6 & -6 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & -3 & -6 & 4 & 9
  \end{bmatrix}
  \]

  • We got an all-zero row in the third row. We exchange this row with the last row.

  \[
  \begin{bmatrix}
  1 & 4 & 5 & -9 & -7 \\
  0 & 2 & 4 & -6 & -6 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & -3 & -6 & 4 & 9
  \end{bmatrix}
  \rightarrow
  \begin{bmatrix}
  1 & 4 & 5 & -9 & -7 \\
  0 & 2 & 4 & -6 & -6 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0
  \end{bmatrix}
  \]
Pivots and rank

• Example cont’d:
  • We continue making elements below the second row pivot $0$.
    \[
    \begin{bmatrix}
    1 & 4 & 5 & -9 & -7 \\
    0 & 2 & 4 & -6 & -6 \\
    0 & -3 & -6 & 4 & 9 \\
    0 & 0 & 0 & 0 & 0
    \end{bmatrix}
    \rightarrow
    \begin{bmatrix}
    1 & 4 & 5 & -9 & -7 \\
    0 & 2 & 4 & -6 & -6 \\
    0 & 0 & 0 & -5 & 0 \\
    0 & 0 & 0 & 0 & 0
    \end{bmatrix}
    \]
  • All elements below the pivot element of the third row is already $0$, and the fourth row does not have any pivot element. So we have now the *echelon* form of the original matrix.
  • The pivots are shown in red, and rank of matrix is $3$. The first, second, and forth column of matrix $A$, which have pivot elements are pivot columns.
    \[
    \begin{bmatrix}
    1 & 4 & 5 & -9 & -7 \\
    0 & 2 & 4 & -6 & -6 \\
    0 & 0 & 0 & -5 & 0 \\
    0 & 0 & 0 & 0 & 0
    \end{bmatrix}
    \]
Free variables

• The elements of $\mathbf{x}$ that are multiplied by non-pivot columns are free variables.

• Example: Find free variables in previous example.
  • In previous example the augmented matrix reduced to its echelon form and we found three pivots. We re-write it in matrix equation form:
    \[
    \begin{bmatrix}
    1 & 4 & 5 & -9 & -7 \\
    0 & 2 & 4 & -6 & -6 \\
    0 & 0 & 0 & -5 & 0 \\
    0 & 0 & 0 & 0 & 0 
    \end{bmatrix}
    \rightarrow
    \begin{bmatrix}
    1 & 4 & 5 & -9 \\
    0 & 2 & 4 & -6 \\
    0 & 0 & 0 & -5 \\
    0 & 0 & 0 & 0 
    \end{bmatrix}
    \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4 
    \end{bmatrix}
    =
    \begin{bmatrix}
    -7 \\
    -6 \\
    0 \\
    0 
    \end{bmatrix}
    \]
  • Re-write the above augmented matrix in the form of vector equation:
    \[
    \begin{bmatrix}
    1 \\
    0 \\
    0 \\
    0 
    \end{bmatrix}
    x_1
    +
    \begin{bmatrix}
    4 \\
    2 \\
    0 \\
    0 
    \end{bmatrix}
    x_2
    +
    \begin{bmatrix}
    5 \\
    4 \\
    0 \\
    0 
    \end{bmatrix}
    x_3
    +
    \begin{bmatrix}
    -9 \\
    -6 \\
    -5 \\
    0 
    \end{bmatrix}
    x_4
    =
    \begin{bmatrix}
    -7 \\
    -6 \\
    0 \\
    0 
    \end{bmatrix}
    \]
  • $x_3$ is a free variable since it is multiplying in the non-pivot column.
Free and basic variables

• The $x_i$’s that are scale factors of non-pivot columns are free variables.
• The $x_i$’s that are scale factors of pivot columns are basic variables.
• Example:
  • In previous example $x_1$, $x_2$, and $x_4$ are basic variables, while $x_3$ is free variable.
  
$\begin{bmatrix}
1 & 4 & 5 & -9 & -7 \\
0 & 2 & 4 & -6 & -6 \\
0 & 0 & 0 & -5 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 4 & 5 & -9 & -7 \\
0 & 2 & 4 & -6 & -6 \\
0 & 0 & 0 & -5 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} =
\begin{bmatrix}
-7 \\
-6 \\
0 \\
0
\end{bmatrix}$

• If matrix $m \times n$ matrix $A$ has rank of $r$, then $A$ has $r$ pivot columns and $n - r$ non-pivot columns.
• In $Ax = b$, the solution, if it exists, has $r$ basic and $n - r$ free variables.

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